

Non-adiabacity and large fluctuations in a many particle Landau Zener problem

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We consider the behavior of an interacting many particle system under slow external driving – a many body generalization of the Landau-Zener paradigm. We find that a conspiracy of interactions and driving leads to physics profoundly different from that of the single particle limit: for practically all values of the driving rate the particle distributions in Hilbert space are very broad, a phenomenon caused by a strong amplification of quantum fluctuations in the driving process. These fluctuations are ‘non-adiabatic’ in that even at very slow driving it is exceedingly difficult to push the center of the distribution towards the limit of full ground state occupancy. We obtain these results by a number of complementary theoretical approaches, including diagrammatic perturbation theory, semiclassical analysis, and exact diagonalization.

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I. INTRODUCTION

In many physical contexts, one is met with quantum systems that are subject to ‘slow’ time dependent external driving. The most elementary prototype in this category, the Landau-Zener (LZ) system [1, 2], contains just two coupled levels driven linearly in time. In this system, the initially occupied instantaneous ground state level of its Hamiltonian

$$\hat{H} = \begin{pmatrix} \lambda t & g \\ g & -\lambda t \end{pmatrix}, \quad (1)$$

stays occupied in the infinite future with a probability

$$P = 1 - e^{-\frac{\pi g^2}{\lambda}}. \quad (2)$$

where g is the coupling strength, λ defines the driving rate and $\exp(\pi g^2/\lambda)$ is the so-called Landau-Zener parameter. The approach $P \xrightarrow{\lambda \rightarrow 0} 1$ is manifestation of the quantum adiabatic theorem, i.e. the statement that sufficiently slow driving keeps a quantum system in its adiabatic ground state.

Many quantum single particle systems can be effectively described in terms of the Landau-Zener setup, or one of its multi-dimensional generalizations [3, 4, 5, 6]. The reason is that the driven approach of pairs of instantaneous eigenstates will generate ‘avoided crossings’ which can be represented by Hamiltonians such as in (1). The cumulative statistics of these crossings then describes the behavior of the system in the course of the driving process. In particular, the system will remain in its ground state if only the latter is sufficiently well separated from the first excited states.

But how do many particle systems behave under driving? Given the exponential abundance of energy levels in interacting systems (or the fact that superimposed on

the groundstate we often have an continuum of low lying, ‘soft’ excitations) reference to the adiabatic theorem will not be sufficient to understand the consequences of driving. In some instances, it is possible to reduce the problem to one of studying the statistics of linear (oscillator) excitations superimposed on an invariant ground state. A system of this type has been studied in work by Yurovsky, Ben-Reuven and Julienne [7] (see also [8]), with the principal observation that the driving process generates a number of x excitations, where

$$x \equiv \exp(\pi g^2/\lambda) \quad (3)$$

coincides with the Landau-Zener parameter.

In general, however, many particle systems cannot be linearized. The ramifications of non-linearities in a driven context have been studied in Ref. [9, 10], within the simplifying framework of a low-dimensional system (specifically, a Bose-condensed system which can be described in terms of a single complex condensate amplitude.) Describing its dynamics in terms of a nonlinear Schrödinger equation, Ref. [10] observed examples of rather interesting behavior, including situations where the system does not remain in the ground state even in the fully adiabatic limit. (For this, however, authors had to consider systems of bosons with attractive interactions. In practice, such systems tend to collapse.) The problem is also particular in that it has a low dimensional Hilbert space.

Refs. [11, 12] considered an interacting driven system which is generic in that it shows the two principal characteristics of interacting quantum systems: a high dimensional Hilbert space and nonlinearity. In one of its representations, the system describes a large assembly of degenerate fermions which may pair-combine into a bosonic level as their energy gets pushed up (a cartoon of a fermion/boson conversion process as realized in, say, a time-dependent BEC/BCS crossover.) Quite strikingly,

it turned out that this system resists getting close to its adiabatic ground state, i.e. a state where all particles eventually have become bosonic; even at very slow driving an $\mathcal{O}(1)$ fraction of particles remains in energetically high-lying sectors of Hilbert space. In fact, the methods employed in Ref. [11] did not enable us to get close to the ‘adiabatic regime’ of the model. The approach to adiabaticity was discussed in Ref. [12] by mapping the evolution of the system to an effective Hamiltonian dynamical system. In this way, as we show in this paper, the exponential dependence (2) gives way to a power law,

$$1 - \frac{n_b}{N} \sim \lambda,$$

controlling the ground state occupancy for a wide range of initial conditions. Here $n_b \in [0, N]$ is the number of particles in the bosonic ground state, and N the total number of particles. Formally, the reluctance of the system to approach its ground state may be understood in terms of dynamical instabilities [12] (to be discussed in some detail further below.) Physically, it reflects the general inertia of interacting systems to adjust to (time dependent) environmental changes.

In this paper, we will discuss an interaction phenomenon which is no less remarkable: the slow dynamical evolution of the system goes along with an exceptionally strong buildup of quantum fluctuations: although we are dealing with a system that should behave ‘semiclassically’, on account of the largeness of its number of particles, N , the distributions of particles in Hilbert space turn out to be very broad. This phenomenon can be attributed to an amplification of quantum fluctuations: classically, the driving process changes the relative energy of two many particle systems. At some point, the originally lower (stable) system becomes higher in energy (unstable). In a strict classical sense, however, the ground state of the elevated system remains stationary. It takes the action of weak ($\mathcal{O}(N^{-1})$) quantum fluctuations to destabilize this state and initiate the evolution towards energetically more favorable sectors of Hilbert space. The nonlinearity of that evolution (interactions) then leads to an amplification of the initial fluctuations, up to a point where they become of $\mathcal{O}(1)$. Specifically, we find that

- in all but the extreme adiabatic limit $N - n_b \ll N$ the ground state occupation n_b shows massive fluctuations, $\text{var}(n_b)/\langle n_b \rangle^2 = \mathcal{O}(1)$. At intermediate mean occupancy $\langle n_b \rangle \simeq N/2$, the distribution $P(n_b)$ extends over almost the full interval $[0, N]$.
- At fast rates $n_b \ll N$, the probability distribution is exponential: $P(n_b) \propto \exp[-n_b/\langle n_b \rangle]$. At slower rates, it becomes even broader and covers Hilbert space in a manner for which we have no analytic expressions. At yet slower rates, upon approaching the adiabatic limit, the distribution gets ‘squeezed’ into the boundary region $n_b/N \sim 1$. It then as-

sumes a universal Gumbel form [13], the latter frequently appearing in the context of extreme value statistics.

In this paper, we will set the stage for the discussion of fluctuations by establishing contact between our earlier ‘quantum’ approach to the problem – the latter adjusted to regimes of fast and moderate driving – and a slow driving semiclassical formulation. Our semiclassical approach is different from that Ref. [12] in that it microscopically connects to the early quantum stages of the evolution, a matching procedure necessary to explore the role of fluctuations. (In fact, we also doubt validity of the results for the *mean* conversion rates derived in that earlier reference: for all but zero initial occupancy of the boson state, a power law in the driving parameter is predicted that we believe is incorrect.)

The rest of the paper is organized as follows: in section II we will introduce the spin variant of the model, and discuss the mapping to its other representations. We will then analyse the system, in a manner that is structured according to the driving rate: in section III we discuss the regime of moderately high driving rates, where the system can be effectively linearized and admits a full analytic solution. In section IV we go beyond the linear regime and consider driving rates $1 < \lambda^{-1} < \mathcal{O}(\ln N)$ in terms of effective rate equations. Although these equations become uncontrolled for values $\lambda^{-1} \sim \mathcal{O}(\ln N)$, they are good enough to signal the system’s reluctance to approach the adiabatic limit. In section V we will formulate the semiclassical approach to the model, and the so-called truncated Wigner approximation. This approximation becomes highly accurate at sufficiently large $N \gtrsim 10$, and for all values of the driving rate. In section VI we use the method of adiabatic invariants to explore the semiclassical theory at very slow driving, $\lambda^{-1} = \mathcal{O}(\ln N)$. In section VII, the quality of the results obtained in this way will be checked by comparison to direct numerical solutions of the Schrödinger equation of our problem (which is feasible for particle numbers up to $N = \mathcal{O}(10^3)$) and to the simulations of the semiclassical approximation (the latter extensible to much larger N .) We will also compare to previous work in the literature. In section VIII we discuss a number of ramifications of our problem relating to previous theoretical and experimental work. We conclude in section IX.

II. THE MODEL

In this section we will introduce the theoretical model discussed in much of the rest of the paper – a high-dimensional generalization of the spin-boson model. We will also introduce a number of less abstract equivalent representations, some resonant with concrete experimental activity.

A. Definition of the model

Consider an SU(2) spin of value $S = N/2 \gg 1$, coupled to a time varying magnetic field of strength $(-\lambda t)$ in z -direction. This will be the first quantum system participating in the driving process. Its partner system is a single bosonic mode at energy $(-\lambda t)$. We couple these two compounds by declaring that the creation of a boson goes along with a lowering of the spin by one. The total system is then described by the Hamiltonian

$$\hat{H} = -\lambda t b^\dagger b + \lambda t S^z + \frac{g}{\sqrt{N}} (b^\dagger S^- + b S^+), \quad (4)$$

where g/\sqrt{N} defines the coupling strength and $S^\pm = S_x \pm iS_y$. The Hamiltonian (4) obeys the conservation law $[\hat{H}, S^z + b^\dagger b] = 0$, showing that the total Hilbert space dimensionality of the problem is $2S + 1 = N + 1$. The linear growth of the dimension of Hilbert space in N is of course not representative for 'generic' interacting systems (dimensionality exponential in N .) An increase in the dimensionality of the problem can be effected by symmetry breaking, e.g., by replacement of the large spin by an assembly of N non-degenerate spin 1/2 compounds (cf. discussion in section II C below.) While this generalization will make the problem largely un-tractable, we believe it to have little qualitative effect, as long as the band-splitting is smaller than the hybridization strength with the boson mode.

Below it will be useful to think of b as a transverse magnetic field acting on the spin. The above conservation then implies that the transverse field strength is proportional to the deviation of the spin off total polarization $S_z = S$. This feedback mechanism of the spin precession into the field strength encapsulates the effect of interactions in the spin variant of our model. However, before proceeding, let us introduce a few alternate representations which make the interpretation of the model as one of interacting particles more transparent (cf. Fig. 2):

1. The variant dominantly discussed in earlier work describes the hybridization of energetically degenerate pairs of spinful fermions with a boson mode – cf. the Hamiltonian (7) below. This may be viewed as a dispersionless approximation of the fermion-boson conversion processes realized in BEC-BCS crossover experiments in fermionic condensates [14, 15], cf. Fig. 2, b). (Although the initial full occupancy of the flat fermion band assumed in AG may not be adequate to the description of the experimental situation [16], see section VIII below.)
2. Identifying the empty (doubly occupied) configurations of the fermion levels with the two states of a

spin 1/2, the system maps onto a time-dependent variant of the Dicke model [17]. In this form it is relevant to the description of super radiance phenomena in molecular magnetism [18], and to cavity QED with many two level systems [19].

3. In a somewhat less obvious representation, the model describes the conversion of pairs of bosons into dimers. In this incarnation it is of relevance to recent experimental work of the JILA group. In these experiments, identical [20] or different [21] species of atoms undergo sweep through a Feshbach resonance to form di-atomic molecules. (On the level of effective classical equations of motions, this correspondence was noted in earlier references[12]. Below, we will establish the connection within a fully quantum mechanical setting.)

For the convenience of interested readers, the equivalence between these different incarnations of the model is established in subsection II C below. While the 'spin-boson' formulation (4) does not relate to concrete physical systems in an obvious way, we find it ideally suited to our theoretical analysis and much of our later discussion will be formulated in this language. However, it is straightforward to transcribe all conclusions to the context of the other models.

B. Formulation of the problem

The problem we will address in much of the paper is formulated as follows. Start the dynamics in the distant past in the adiabatic ground state of the problem: spin fully polarized $S_z = S = N/2$ and zero bosons $n_b(t) \equiv \langle b^\dagger(t)b(t) \rangle = 0$. The goal is to find the number of produced bosons at large positive times,

$$n_b \equiv \lim_{t \rightarrow \infty} n_b(t) \equiv \lim_{t \rightarrow \infty} \langle b^\dagger(t)b(t) \rangle. \quad (5)$$

The adiabatic limit is reached when $n_b \rightarrow N$ or, equivalently, $S_z = -S$. Alternatively, we can speak of the representation in terms of bosonic atoms and bosonic molecules, given by Eqs. (8) and (10), where this initial condition implies the absence of atoms in the distant past, and the goal is to find the number of produced atoms at large positive times.

As was anticipated in the introduction, the operator $b^\dagger b$ exhibits strong quantum fluctuations. It will, thus, be of interest to explore the *distribution* $P(n)$ defined by the moments $\langle (b^\dagger b)^k \rangle$. Also, there may be physical situations where it is not appropriate to start the process in the spin-up polarized state. We have already mentioned the case of dilute fermion gases [16] where a near south polar configuration may be a more appropriate starting point. Similarly, a downward polarized state $S_z = -S$ (in an initially downward pointing field) will model an atom \rightarrow molecule conversion process with a purely atomic starting configuration.

In the following, we will describe different stages of the conversion process, where the organizing principle will be the conversion efficiency, which in turn depends on the speed of the sweeping process. For later reference, the different parameter regimes, along with the theoretical methods we will use to describe them, are summarized in Fig. 1.

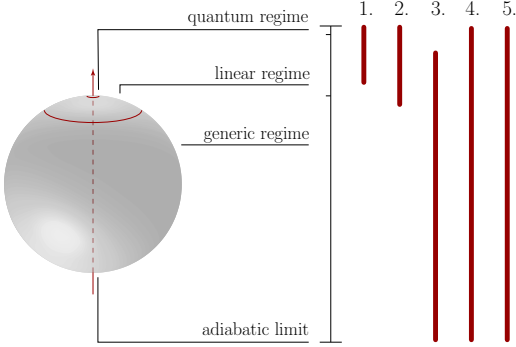


FIG. 1: Different theoretical approaches to the problem. The Hilbert space of the problem contains a number of different sectors: a (quantum fluctuation) dominated region where only a few bosons have been created $n_b = \mathcal{O}(1)$ and the spin remains nearly polarized, a 'linear region', where the number of bosons may be large but is small enough to justify linearization of spin operators, $1 \ll n_b \ll N$, a 'generic' region, $n_b = \mathcal{O}(N)$, and the adiabatic limit $n_b \rightarrow N$. Theoretical approaches we will use to approach these regimes include, 1.) solution of the linearized Schrödinger equation (cf. section III), 2.) Keldysh perturbation theory based on a self consistent RPA approximation (cf. section IV), 3.) semiclassical approach based on the approximate conservation of adiabatic invariants (cf. section VI), 4.) an extended semiclassical scheme, where semiclassical methods are employed to propagate the full quantum distribution beyond the quantum regime (cf. sections V and VII), and 5.) numerical solution of the Schrödinger equation for moderately large N .

C. Alternate model representations

The equivalence to a system of boson-coupled degenerate two level systems – the *time dependent Dicke model* –

Noting that the interaction couples only two ($|\uparrow, \downarrow\rangle$ and $|0\rangle$) of four possible ($|\uparrow, \downarrow\rangle$, $|\uparrow\rangle$, $|\downarrow\rangle$ and $|0\rangle$) occupation states of a spinful fermion level, we may introduce a pseudospin state to discriminate between these two states, $|\uparrow, \downarrow\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|0\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The pseudospin

is best seen starting from the latter. Consider the Hamiltonian

$$\hat{H} = -\lambda t b^\dagger b + \frac{\lambda t}{2} \sum_{i=1}^N \sigma_i^z + \frac{g}{\sqrt{N}} \sum_{i=1}^N (b^\dagger \sigma_i^- + b \sigma_i^+), \quad (6)$$

where $\sigma^{x,y,z}$ are Pauli matrices acting in pseudospin space. The fact that the spin operators appear in totally symmetric combinations, $\sum_i \sigma_i^{x,y,z}$ suggests introducing the total spin operator $S^{x,y,z} \equiv \frac{1}{2} \sum_i \sigma_i^{x,y,z}$. The operators $S^{x,y,z}$ define an SU(2)-algebra. Denoting the

maximum weight space 'all fermions occupied' by $|S\rangle$, we observe that S^z has maximal eigenvalue $S = N/2$, $S^z|S\rangle = S|S\rangle$. This shows that the operators $S^{x,y,z}$ act in a Hilbert space of dimension $2S + 1 = N + 1$. The global spin representation of the Hamiltonian (6) is but the starting Hamiltonian (4). The Hamiltonian Eq. (6) is represented on Fig. 2 a). As mentioned in the Introduction, a natural experimental realization of Eq. (6) is a cavity QED with N two-level systems.

The *fermi-bose crossover model* describes a system of $2N$ degenerate spinful fermion states. Two fermions occupying such a state may combine to form a boson populating one ('condensate') bosonic state. Starting from an infinitely low initial value, the energy of the fermions gets pushed up linearly in time (cf. Fig. 2, b)). This setup is described by the Hamiltonian

$$\hat{H} = -\lambda t b^\dagger b + \frac{\lambda t}{2} \sum_{i=1}^N (a_{i\uparrow}^\dagger a_{i\uparrow} + a_{i\downarrow}^\dagger a_{i\downarrow}) + \frac{g}{\sqrt{N}} \sum_{i=1}^N (b^\dagger a_{i\downarrow} a_{i\uparrow} + b a_{i\uparrow}^\dagger a_{i\downarrow}^\dagger), \quad (7)$$

where λ is the driving rate, g/\sqrt{N} the coupling amplitude of the conversion process, and $a_{i,\uparrow\downarrow}$ and b are fermion and boson annihilation operators. For convenience, we have split the time dependent energy symmetrically between bosons and fermions. (By a gauge transformation any distribution of energies with fixed difference $2\lambda t$ between a boson and a pair of fermions can be realized.)

representation of the Hamiltonian is but (6).

We may think of the Hamiltonian (7) as a cartoon version of more realistic models. For example, it can be interpreted as a non-dispersive (flat-band) approximation to a system of free fermions coupled to the lowest

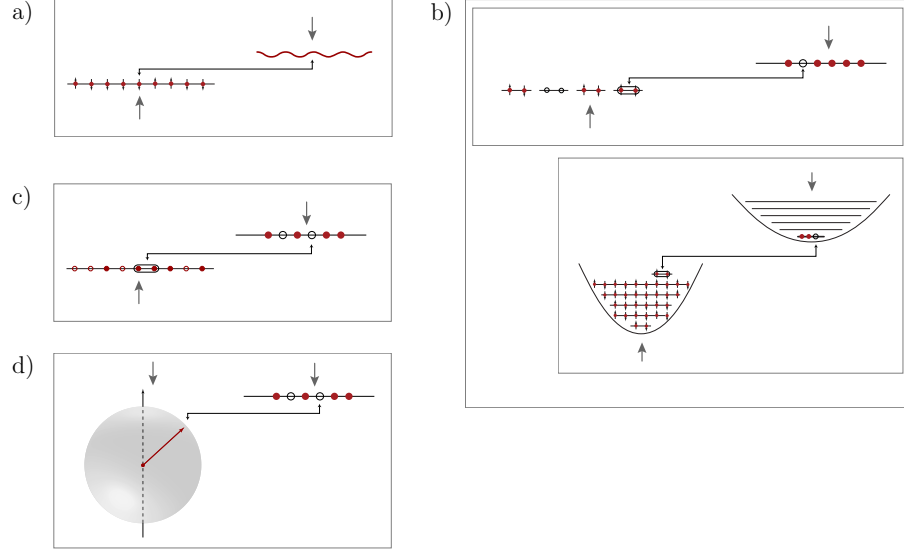


FIG. 2: The different incarnations of the model: a) time dependent Dicke model: an assembly of degenerate two level systems coupled to a boson mode, b) flat band approximation (top) to a system of spinful fermions pair-binding into bosonic ‘molecules’, c) system of atoms pair-converting into molecules, d) spin subject to a magnetic field and coupled to a bosonic level.

mode of an electromagnetic field in a cavity QED setup. Alternately, it may be regarded as a cartoon version of a BCS/BEC crossover system wherein a band of (lattice) fermions is initially filled, the band dispersion is neglected, and only the lowest condensate state of the bosons (‘molecules’) is kept (Fig. 2 b) inset). While the neglect of dispersion effects may be acceptable [16], the assumption of full band occupancy might not be met in experiment. (Ref. [16] argues that the full band assumption should be modeled by starting the dynamics in incompletely ‘polarized’ (pseudo)spin states, and we refer the reader to this work for a discussion of this case.) For completeness, let us mention that the problem of the dynamic BCS-BEC crossover has been addressed in many publications and using different approaches, beginning with [22] and continuing with Refs. [12, 16, 23, 24, 25]. It is probably fair to say that a consensus on whether this problem can indeed be reduced to the flat-band model with a single Bose mode Eq. (7) has not yet been reached in the literature.

Finally, consider a system of atoms sweeping through a Feshbach resonance whereafter di-atomic molecules form the energetically preferred state (cf. Ref. [20] where the conversion processes has been realized in condensates of ^{85}Rb or ^{40}K atoms.) This situation is described by the Hamiltonian

$$\hat{H} = \lambda t d^\dagger d - \frac{\lambda t}{2} c^\dagger c + \frac{g}{\sqrt{N}} (d^\dagger c c + d c^\dagger c^\dagger). \quad (8)$$

This Hamiltonian describes the *conversion of atoms to molecules*, where the former/latter are created by the

bosonic operators c^\dagger/d^\dagger , and the symmetry $[\hat{H}, d^\dagger d + 2c^\dagger c] = 0$ reflects the conservation of the total number of particles. The time-independent version of this model was extensively analyzed in Ref. [26].

The (near) equivalence to the previous model representations is seen by computing matrix elements $M_{N,n} \equiv \langle N-n+1, 2n-2 | \hat{H} | N-n, 2n \rangle$, where the state $|N-n, 2n\rangle$ contains $(N-n)$ d -molecules and $2n$ c -atoms. Evaluating the matrix element, we find

$$M_{N,n} = \frac{g}{\sqrt{N}} \sqrt{(N-n+1)n(n-1/2)}.$$

Now, let $|N/2 - n, n\rangle'$ be the state with n bosons and spin polarization $S_z = N/2 - n$. Computing the matrix elements $M'_{N,n} \equiv \langle N/2 - n + 1, n - 1 | \hat{H} | N/2 - n, n \rangle$ of the Hamiltonian (4), we find

$$M'_{N,n} = \frac{g}{\sqrt{N}} \sqrt{(N-n+1)n^2}. \quad (9)$$

The matrix elements M and M' are identical, up to corrections of $\mathcal{O}(n^{-1})$ which vanish in the limit of large particle number occupancies. This consideration shows that the conversion of atoms to molecules can be discussed within the framework of the spin Hamiltonian above, where the number of molecules is counted in terms of the spin quantum number.

The correspondence between models can be made perfect by considering heteromolecular conversion processes (cf. Ref. [21] for a realization in an $^{85}\text{Rb} - ^{87}\text{Rb}$ gas.) Straightforward generalization of (8) obtains the Hamil-

model	system A	system B
spin-boson	spin $S = N/2$	boson mode
time dependent Dicke model	N two-level systems	boson mode
fermi-bose conversion	flat band of $2N$ fermion states	boson mode
atom-molecule conversion	di-bosonic molecular state	boson mode

TABLE I: survey of the equivalent representations of the model

tonian

$$\hat{H} = \lambda t d^\dagger d - \frac{\lambda t}{2} (c_1^\dagger c_1 + c_2^\dagger c_2) + \frac{g}{\sqrt{N}} (d^\dagger c_1 c_2 + d c_1^\dagger c_2^\dagger), \quad (10)$$

where $c_{1/2}^\dagger$ creates atoms of species 1/2 and d^\dagger the (1-2) heteromolecule. It is not difficult to verify that the matrix elements of this Hamiltonian *exactly* coincide with

III. LINEARIZED DIABATIC REGIME

We first consider moderately fast driving rates where only a small fraction of the particles undergoes a conversion process. In the language of the spin model this means that at large times the spin will still be stuck close to the north polar regions, $\lim_{t \rightarrow \infty} (S - S_z(t))/S \ll 1$, or $n_b/N \ll 1$. In this regime, the curvature of the SU(2) spin manifold is not yet felt and the problem can be effectively linearized.

This is best done in the Holstein-Primakoff representation of spin operators

$$\begin{aligned} S^- &= c^\dagger (2S - c^\dagger c)^{1/2}, \\ S^+ &= (2S - c^\dagger c)^{1/2} c, \\ S^z &= S - c^\dagger c, \end{aligned} \quad (11)$$

where the algebra of bosonic operators, c and c^\dagger defines the Holstein-Primakoff bosons. Substitution of the linearized approximation $S^- = (2S)^{1/2} c^\dagger$, $S^+ = (2S)^{1/2} c$, $S^z = S - c^\dagger c$ into (4) generates the quadratic Hamiltonian

$$\hat{H}_{\text{HP}} = -\lambda t (b^\dagger b + c^\dagger c) + g(b^\dagger c^\dagger + cb). \quad (12)$$

To compute the number of bosons created in the linearized evolution, we switch to the Heisenberg representation $b(t) = e^{i\hat{H}t} b e^{-i\hat{H}t}$ and consider the expectation value

$$n_b(t) = \langle b^\dagger(t) b(t) \rangle,$$

taken with respect to the vacuum state, $\langle \dots \rangle \equiv \langle 0 | \dots | 0 \rangle$, with $b|0\rangle = c|0\rangle = 0$. (Alternatively, one can consider

those of the spin-boson Hamiltonian (4).

In this incarnation our problem has been analyzed by a number of authors, beginning from Ref. [7] (see also the more recent Ref. [27]). It is also in this form that the problem is the closest to existing experiments [20, 21].

Eqs. (4), (6), (7), and (8) define four equivalent definitions of the model. For the convenience of the reader, the correspondence between the systems involved in these definitions is summarized in table I.

a model of bosonic atoms and molecules, in the regime where most of the particles are molecules. In the Hamiltonians (8) and Eq. (10) one may then replace $d \rightarrow \sqrt{N}$, to arrive precisely at Eq. (12), an appropriate relabeling of operators understood. In Ref. [7] a model equivalent to the linearization above was introduced, and the mean conversion rates of the linearized dynamics were computed (see also Ref. [8]).

The computation of $n_b = \lim_{t \rightarrow \infty} n_b(t)$ is reviewed in Appendix A, with the principal result

$$n_b = x - 1. \quad (13)$$

Eq. (13) states that the number of converted particles grows linearly in the LZ parameter. Unlike with the standard (two-level) LZ problem, the many particle system remains parametrically detached from adiabaticity, even for large values of the LZ parameter. However, the linearization underlying the derivation limits the validity of the result to $n_b \ll N$.

The techniques used in Appendix A actually yield the entire (quantum) distribution of the number of produced particles,

$$P(n) = \frac{1}{x} (1 - x^{-1})^n \stackrel{x \gg 1}{\simeq} \frac{e^{-n/x}}{x}. \quad (14)$$

This distribution is very broad. Its width $(\langle n^2 \rangle - \langle n \rangle^2)^{1/2} \simeq x$ is of the same order as its mean. This is remarkable inasmuch as our system is not subject to any sources of fluctuations, other than its intrinsic quantum fluctuations. We are met with the unusual situation that quantum fluctuations are of the same order as mean values, in spite of the 'semiclassical' largeness of the latter, $\langle n \rangle \simeq x \gg 1$. Similar phenomena have been discussed in the context of dynamical sweeps through quantum phase transition points, cf. e.g. Ref. [28].

IV. THE KINETIC EQUATION

A. Keldysh formalism

An alternative way of analyzing the problem starts from the boson-fermion representation (7) and analyzes the corresponding kinetic equation – theory strand b) in Fig. 2. To formulate this approach, we employ the Keldysh formalism and introduce the appropriate Keldysh Greens functions (cf. Ref. [29] for a review of the formalism.) Readers not interested in the formalism are invited to proceed directly to the discussion of the main result of this section, the rate equation (27).

Within the Keldysh formalism, the fermion subsystem is described by the matrix Greens function

$$G_{\sigma\alpha,\sigma'\alpha'}(t,t') = -i \left\langle a_{\sigma\alpha}(t) a_{\sigma'\alpha'}^\dagger(t') \right\rangle = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}_{\sigma\alpha,\sigma'\alpha'}(t,t'),$$

where $\alpha, \alpha' = 1, 2$ index the two-component Keldysh space (to which the matrix structure in the equation refers), and $\sigma = (i, \uparrow\downarrow)$ is a container index comprising the fermion site and spin index. G^R , G^A , and G^K stand, respectively, for the advanced, retarded and Keldysh Greens function. The latter is defined by

$$G^K = G^R \circ f - f \circ G^A,$$

where the symbol ' \circ ' defines a temporal convolution,

$$(A \circ B)(t, t') \equiv \int dt'' A(t, t'') B(t'', t')$$

and $f \equiv 1 - 2n_f$ defines the distribution of the fermions. In what follows, it will be convenient to represent the Greens functions of the theory in the Wigner representation,

$$G^K(\tau, \omega) = \int dt G^K \left(\tau + \frac{t}{2}, \tau - \frac{t}{2} \right) e^{i\omega t}, \quad (15)$$

and similarly for all other Green functions. In the absence of interactions, $g = 0$, a straightforward calculation obtains the Wigner representation of the free Greens functions as

$$G_{0,\sigma\sigma'}^R(\tau, \omega) = \frac{\delta_{\sigma\sigma'}}{\omega^+ - \lambda t/2}, \quad (16)$$

$$G_{0,\sigma\sigma'}^A(\tau, \omega) = \frac{\delta_{\sigma\sigma'}}{\omega^- - \lambda t/2},$$

$$G_{0,\sigma\sigma'}^K(\tau, \omega) = -2\pi\delta_{\sigma\sigma'} f_0(\tau, \lambda t/2) \delta(\omega - \lambda t/2),$$

where $f_0(\tau, \lambda t/2) \equiv f_0(\tau)$ describes the distribution of the full fermion band, $f_0(\tau) = 1 - 2n_0(\tau) = -1$. In a similar manner the bosons are described by

$$D(t, t') = -i \langle b_\alpha(t) b_{\alpha'}^\dagger(t') \rangle = \begin{pmatrix} D^K & D^R \\ D^A & 0 \end{pmatrix}_{\alpha\alpha'}(t, t'), \quad (17)$$

where

$$D^K = D^R \circ F - F \circ D^A,$$

and $F = 1 + 2n_b$, n_b describing the boson distribution. The Wigner representation of the non-interacting boson Greens functions is given by

$$\begin{aligned} D_0^R(\tau, \omega) &= \frac{1}{\omega^+ + \lambda t}, \\ D_0^A(\tau, \omega) &= \frac{1}{\omega^- + \lambda t}, \\ D_0^K(\tau, \omega) &= -2\pi F_0(\tau, -\lambda t) \delta(\omega + \lambda t), \end{aligned} \quad (18)$$

where $F_0(\tau, -\lambda\tau) \equiv F_0(\tau)$ relates to the vanishing boson number of the non-interacting problem as $F_0(\tau) = 1 + 2n_0(\tau) = 1$.

In order to calculate the number of produced bosons at the end of the Landau-Zener process, we need to calculate the exact bosonic Keldysh Greens function, in terms of which

$$\begin{aligned} n_b &= \lim_{t \rightarrow \infty} n_b(t) = \lim_{t \rightarrow \infty} (iG^K(t, t) - 1) / 2 \simeq \\ &\simeq \frac{1}{2} \left(\lim_{\tau \rightarrow \infty} \int \frac{d\omega}{2\pi} F(\tau, \omega) A_b(\tau, \omega) - 1 \right), \end{aligned} \quad (19)$$

where $A_b = -2\text{Im } D^R$ is the spectral function of the bosons. For large positive times, the Greens functions become effectively uncoupled, $D^R(\omega)$ asymptotes to the free form (18) and $A_b(\omega) \simeq 2\pi\delta(\omega + \lambda t)$. This means that the number of bosons at the end of the process is given by

$$n_b = \lim_{\tau \rightarrow \infty} \frac{F(\tau, -\lambda\tau) - 1}{2}. \quad (20)$$

B. Perturbation theory

First, we can try to calculate the bosonic Greens function perturbatively, in powers of g (which means, in practice, in powers of the dimensionless parameter g^2/λ). This expansion is valid at fast rate λ , $g^2/\lambda \ll 1$. The lowest order correction to the bosonic propagator is depicted in Fig. 3, diagram a). The calculation of this diagram, although cumbersome, proceeds in a straightforward fashion, giving the result

$$n_b \approx \frac{\pi g^2}{\lambda}. \quad (21)$$

Each vertex of the diagram carries a factor of g/\sqrt{N} . However, there are N fermions propagating around the loop, canceling the factor $1/N$ coming from the vertices. This results in the cancellation of factors of N in the answer.

In the next order of perturbation theory two more diagrams contribute to the bosonic propagator. These are

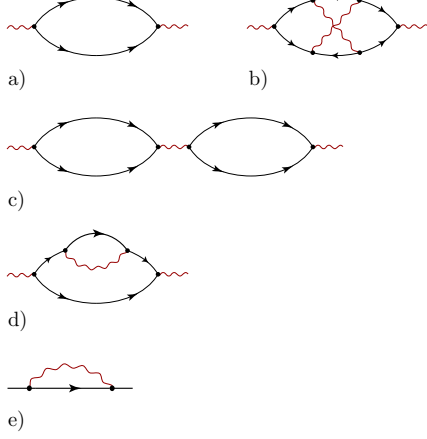


FIG. 3: The lowest order corrections to the bosonic propagator.

shown on Fig. 3, diagrams c) and d). This leads to the second order answer

$$n_b \approx \frac{\pi g^2}{\lambda} + \left(\frac{1}{2} - \frac{1}{N} \right) \left(\frac{\pi g^2}{\lambda} \right)^2. \quad (22)$$

Note that at $N = 1$, this indeed coincides with the Taylor expansion of the exact Landau-Zener formula

$$n_b = 1 - e^{-\frac{\pi g^2}{\lambda}}. \quad (23)$$

Notice also that at $N > 1$, the number of produced particles is bigger than at $N = 1$. This is natural, as with larger N one can produce more bosons.

Finally, we note that the calculation leading to Eq. (22) closely parallels that employed in Ref. [24] to calculate the molecule production in a Feshbach resonance experiment perturbatively.

C. Self consistent RPA and kinetic equation

Our aim now is to calculate the number of produced bosons in the large N limit, with all other parameters

including g and λ kept fixed. Our starting point is the system of kinetic equations (the Wigner transforms of the Dyson equations, cf. Ref.[29]) for the boson and fermion distribution function,

$$(\partial_\tau - \lambda \partial_\omega) F(\tau, \omega) = i (\Sigma^K - (\Sigma^R \circ F - F \circ \Sigma^A)) (\tau, \omega), \quad (24)$$

$$\left(\partial_\tau + \frac{\lambda}{2} \partial_\epsilon \right) f(\tau, \epsilon) = i (\sigma^K - (\sigma^R \circ f - f \circ \sigma^A)) (\tau, \epsilon), \quad (25)$$

The first of these equations is obtained by evaluating the Wigner transform of the general kinetic equation of a boson system [29] $(-i\partial_\tau + [\hat{H}, \cdot])F = \Sigma^K - (\Sigma^R \circ F - F \circ \Sigma^A)$ for the Hamiltonian $\hat{H} = -\lambda t$. The second equation is its fermionic analog.

In the 'collision integrals' of (24), $\Sigma^{K,R,A}(\sigma^{K,R,A})$ are the components of the bosonic/fermionic self energies. It is not difficult to see that the dominant contributions to these self energies are given by the diagrams Fig. 3 a) and e), external legs truncated. Unlike with the lowest order perturbation theory discussed above, the fermionic and bosonic propagators appearing in these diagrams are to be understood as dressed propagators, containing self energy insertions by themselves. (Technically speaking, this means that we will evaluate the self energies in a self consistent RPA approximation.) Self energy diagrams with crossing interaction lines, such as Fig. 3 b), carry factors of N^{-1} relative to the self consistent RPA diagrams. (See, however, the discussion in the end of this section.)

The derivation of the self energies for the boson-fermion interaction follows standard procedures [29] and we will not repeat it here. As a result we obtain the equations

$$\begin{aligned} (\partial_\tau - \lambda \partial_\omega) F(\omega) &= \frac{g^2}{2} \int \frac{d\epsilon}{2\pi} A \left(\omega - \epsilon - \frac{\lambda\tau}{2} \right) A \left(\epsilon - \frac{\lambda\tau}{2} \right) [1 + f(\omega - \epsilon)f(\epsilon) - (f(\epsilon) + f(\omega - \epsilon)) F(\omega)], \\ \left(\partial_\tau + \frac{\lambda}{2} \partial_\epsilon \right) f(\epsilon) &= \frac{g^2}{2N} \int \frac{d\omega}{2\pi} A \left(\omega - \epsilon - \frac{\lambda\tau}{2} \right) A_b(\omega + \lambda\tau) [1 + f(\omega - \epsilon)f(\epsilon) - (f(\epsilon) + f(\omega - \epsilon)) F(\omega)], \end{aligned} \quad (26)$$

where we suppressed the explicit τ -dependencies for no-

tational clarity and $A = -2\text{Im} G^R$ and $A_b = -2\text{Im} D^R$

are the spectral functions of the fermions and the boson, respectively. Notice the factor of N^{-1} multiplying the collision integral of the fermions. The absence of this factor in the boson collision integral signals that the boson interacts with all N fermions simultaneously. In contrast, each fermion interacts only with a single boson, which means that the fermionic self energies carry the uncompensated squared coupling constant g^2/N .

Eq. (26) defines a particle number conserving system. Consider the quantity $N(\tau) \equiv n_b(\tau) + Nn_f(\tau)$. Using Eq. (19), and its fermionic analog $n_f(\tau) = -\frac{1}{2} \left(\int \frac{d\epsilon}{2\pi} f(\tau, \epsilon) A(\tau, \epsilon) - 1 \right)$, we obtain that the number of particles varies in self consistent RPA as

$$\frac{dN}{d\tau} = \frac{1}{2} d_\tau \left(\int \frac{d\omega}{2\pi} F(\tau, \omega) A_b(\tau, \omega) - N \int \frac{d\epsilon}{2\pi} f(\tau, \epsilon) A(\tau, \epsilon) \right).$$

Integrating the first (second) equation (26) over $\int \frac{d\omega}{2\pi} A_b(\tau, \omega)$ ($\int \frac{d\epsilon}{2\pi} A(\tau, \epsilon)$) and using that $\lambda \partial_\omega A_b = \partial_\tau A_b$, we find that the two integrals cancel out, and $d_\tau N = 0$.

This is about as far as we will get without further approximations. Below we will argue that the validity of the RPA approximation is limited by the condition $n_b \ll N$. In the limit $n_b/N \rightarrow 0$, the self energies $\sigma^{R/A}$ broadening the energy dependence of the spectral functions become vanishingly small. (The detailed dependence of the broadening depends on the value of time, τ .) Relying

on this limiting behaviour, we will approximate the spectral functions by $A(\epsilon) \simeq 2\pi\delta(\epsilon)$ by δ -functions. This is a bold approximation inasmuch as the pairs of δ -functions in (26) enforce the resonance condition $\tau = 0$. However, it is this 'resonant' time window, $\tau \simeq 0$ where the distribution functions entering the integral kernel are expected to vary strongest. This means that the value of the collision integral may well be sensitive to the detailed temporal profile of the distribution functions around $\tau = 0$. Ultimately, we will need to compare to the result of other methods to verify the validity of the δ -function approximation.

We substitute the approximations $A_b(\tau, \omega) \simeq 2\pi\delta(\omega + \lambda\tau)$ and $A(\tau, \epsilon) \simeq 2\pi\delta(\epsilon - \lambda\tau/2)$ into (26), use $n_b(\tau) = \frac{1}{2} \left(\int \frac{d\omega}{2\pi} F(\tau, \omega) A_b(\tau, \omega) - 1 \right) \simeq (F(\tau, -\lambda\tau) - 1)/2$, and differentiate w.r.t. time to obtain

$$\partial_\tau n_b = -\partial_\tau n_f = \frac{\pi g^2}{\lambda} \delta(\tau) [n_f^2 + n_b(2n_f - 1)]. \quad (27)$$

The meaning of this equation becomes transparent upon rewriting the combination of distribution functions on its r.h.s as $(1 + n_b)n_f^2 - n_b(1 - n_f)^2$. This factor weighs the probability that two fermions convert into bosons and back. To solve the equation, we take advantage of the conservation law

$$n_b + Nn_f = N. \quad (28)$$

It is then straightforward to obtain the solution

$$n_b(\tau) = \frac{2N \left(e^{\sqrt{\frac{N+4}{N}} \frac{\pi g^2}{\lambda} \theta(\tau)} - 1 \right)}{e^{\sqrt{\frac{N+4}{N}} \frac{\pi g^2}{\lambda} \theta(\tau)} \left(-N + \sqrt{N(N+4)} + 2 \right) + N + \sqrt{N(N+4)} - 2}. \quad (29)$$

The singular behaviour of n_b at $\tau = 0$ is a remnant of the above δ -function approximation. Incidentally, we note that the Taylor series expansion of n_b with respect to g^2 gives

$$n_b = \lim_{\tau \rightarrow \infty} n_b(\tau) = \frac{\pi g^2}{\lambda} + \left(\frac{1}{2} - \frac{1}{N} \right) \left(\frac{\pi g^2}{\lambda} \right)^2 + \frac{6 - 8N + N^2}{6N^2} \left(\frac{\pi g^2}{\lambda} \right)^3 + \dots \quad (30)$$

The first two terms indeed match the direct perturbative expansion given by Eq. (22), confirming that this technique does take into account all the diagrams up to the second order. (In the context of perturbation theory, the δ -function approximation of the spectral functions amounts to the lowest order in g Born approximation to the self energy operator.)

On the other hand, taking the limit $N \rightarrow \infty$ gives

$$n_b = \frac{N \left(e^{\frac{\pi g^2}{\lambda}} - 1 \right)}{2e^{\frac{\pi g^2}{\lambda}} + N}. \quad (31)$$

This is the main result of the kinetic equation approach. For $e^{\pi g^2/\lambda} \ll N$ we find

$$n_b = e^{\frac{\pi g^2}{\lambda}} - 1. \quad (32)$$

This matches the large N limit given by Eq. (13).

The kinetic equation produces the boson number (31) which is monotonously increasing with decreasing λ , until it reaches its limiting small λ value of $N/2$. The reason for this behaviour is that at large N , the r.h.s. of the kinetic equation approaches zero at $n_f = 1/2$. In other words, the rate of fermions converting into bosons is matched by the rate of bosons converting into fermions.

However, we need to remember that at $n_b = N/2$ we are well outside the regime of validity of the approximated kinetic equation. In fact, we are outside the limit of the RPA as such. This is because at $n_b \simeq N$ diagrams that are nominally small in N^{-1} become sizeable, on account of the large value of the distribution function $n_b \simeq N$. For example, it is straightforward to verify that its crossing interaction lines make diagram Fig. 3 b) small in N^{-1} , but this smallness gets counteracted by n_b -dependent propagators in the center regions of the diagram. The bottom line is that the RPA is controlled only in the limit $n_b/N \rightarrow 0$. Our comparison with numerics below will show that it works remarkably well beyond its nominal limits, i.e. the profile (31) represents a good approximation even at $n_b/N = \mathcal{O}(1)$. However, in order to advance into the adiabatic regime, $n_b \rightarrow N$, we need to employ different methods.

V. SEMICLASSICAL ANALYSIS.

Our goal now is to understand the behavior of n_b for arbitrary driving rates λ . We will use that for large $N \gg 1$ our problem approaches a well defined classical limit where both spin and the bosonic fields can be treated as classical objects. The classical equations of motion, obtained by replacing operators in the Hamiltonian (4) by c -numbers and commutators by Poisson brackets, read

$$\begin{aligned} i\dot{b} &= -\lambda tb + \frac{g}{\sqrt{N}} S^-, \\ \dot{S}_z &= -i \frac{g}{\sqrt{N}} (b S^+ - b^* S^-), \\ \dot{S}_x &= -i \frac{g}{\sqrt{N}} (b^* - b) S_z - \lambda t S_y \\ \dot{S}_y &= -\frac{g}{\sqrt{N}} (b + b^*) S_z + \lambda t S_x. \end{aligned} \quad (33)$$

Within the strictly classical setting, these equations have to be solved for the initial conditions: $b(t_0) = 0$, $S_z(t_0) = N/2$, $S_x(t_0) = S_y(t_0) = 0$, where $t_0 \rightarrow -\infty$ is the initial time. It is straightforward to verify that the solution corresponding to these initial conditions reads $(b(t), S_z(t)) = (b(t_0), S_z(t_0)) = (0, N_0)$, i.e. the initial configuration of zero bosons represents a classically stationary (if unstable) solution.

The prediction that no bosons will be generated, is clearly wrong. However, the purely classical treatment has yet another, and related, drawback: for given initial conditions the uniqueness of the classical solution does not permit the buildup of fluctuations. This is in contradiction with our earlier findings that the distribution of n_b is wide at least at fast and intermediate rates.

It is thus obvious that an meaningful description of the classical limit must account for the presence of quantum fluctuations. These fluctuations will lead to an initial destabilization of the configuration $(b(t), S_z(t)) = (0, N_0)$. The nonlinearity of the classical Hamiltonian,

then amplifies the fluctuations and drives the system away from its initial state. The expansion of dynamics in quantum fluctuations for bosons was analyzed in detail in Ref. [30]. Using a path integral construction similar to the Keldysh technique it was shown that to leading order in quantum fluctuations the classical (Gross-Pitaevskii) equations of motion do not change. However, the initial conditions become randomly distributed with the weight given by the Wigner transform of the density matrix. This so called Truncated Wigner Approximation (TWA) originated in quantum optics [31, 32] and has later been adopted to cold atom systems [33]. By now the TWA has become a standard tool in describing the dynamics of cold atoms in regimes where classical Gross-Pitaevskii equations do not suffice (see Ref. [34] for review).

The TWA is most conveniently adopted to our model by using the Schwinger boson representation of spins. The small parameter of this expansion is $1/S$ where $S \rightarrow \infty$ describes the classical spin. In Appendix B we give some details of this derivation. The net result is that the classical equations of motion (33) need to be supplied by stochastic initial conditions distributed according to the Wigner function. For large spin (pointing along z -direction) and vacuum state of bosons, representing the initial conditions in our problem, this Wigner function is approximately given by

$$W(n_b, S_z, S_\perp) \approx \frac{2}{\pi S} \exp[-2n_0] \exp\left[-\frac{S_\perp^2}{S}\right] \delta(S_z - S), \quad (34)$$

where $S_\perp = \sqrt{S_x^2 + S_y^2}$ and n_0 is the initial boson number. The fact that S_\perp and n_0 are not exactly equal to zero is the result of vacuum quantum fluctuations. In this work we are primarily interested in the number of bosons created during the dynamical process, and its fluctuations. The connection between moments n_0^l , $l = 1, 2, \dots$ and the variables b and b^\dagger is established by the Wigner transform identities summarized in Appendix B. For instance,

$$n_0 \rightarrow b^* b - 1/2, \quad n_0^2 \rightarrow (b^* b)^2 - b^* b. \quad (35)$$

(The correction term $b^* b$ appearing in the expression for the second moment ensures that zero point fluctuations do not affect the expectation value of n_0 and n_0^2 , which should be zero in the vacuum state. In the classical limit $n_0 \gg 1$, these terms become inessential.)

By way of an illustration, let us outline how the TWA reproduces the results of the quantum calculation in the linearizable regime. For quadratic Hamiltonians the TWA is exact [30, 31] and we must be able to reproduce all the results of Sec. III. Upon leaving the linear regime, the number of bosons $n(t) = \langle b^\dagger(t) b(t) \rangle$ has become large and semiclassical methods are expected to work with high accuracy. In other words, the exactness of the TWA to the linear regime entails its applicability to the full parameter space of our problem.

In the linear regime, we may assume approximate constancy of S_z : $S_z \approx S = N/2$. Eq. (33) then reduces

to

$$\begin{aligned} i\dot{b} &= -\lambda tb + gs^- \\ i\dot{s}^- &= -gb + \lambda ts^-, \end{aligned} \quad (36)$$

where we introduced $s^- = S^-/\sqrt{N}$. These equations are identical to an exact reformulation of the Schrödinger equation (cf. Eq. (A1) in Appendix A), a consequence of the linearity of the problem. In Appendix B3 we show that the solution of these equations, with account for quantum fluctuations in the initial data, indeed reproduces the results of the full quantum calculation. In particular we show there that $n_b = x - 1$ and $\langle n_b^2 \rangle = 2x^2 - 3x + 1$ (here as before, we reserve the notation n_b for the number of bosons at the end of the process, as in Eq. (5)). Both results perfectly agree with the exact distribution (14).

At slow values of driving, $x \sim N$ the nonlinear equations (33) no longer afford an exact solution. In the next section we will discuss an approximate solution scheme, based on the method of adiabatic invariants. In Sec. VII we will analyze in detail $n_b(\lambda)$ and its distribution in various regimes using both exact and TWA simulations.

VI. DEEP ADIABATIC LIMIT

A. Adiabatic Hamiltonian

In this section, we will apply the concept of classical adiabatic invariance to the adiabatic limit of the driving process. Considering a fixed initial value $n_0 \sim 1$ (consistent with Eq. (34)), we will be sloppy about initial conditions. This is good enough to obtain reliable results for the mean value of produced bosons, $\langle n_b \rangle$, but won't suffice to obtain the statistics of the driving process. The latter will be explored in section VII below within the more sophisticated TWA scheme.

The concept of the adiabatic evolution of classical invariants [35] was previously applied to the semiclassical limit of a non-linear Schrödinger equation [10] and later [12] to the classical equations of our model. However, (for all but very small, compared with N , initial occupancy $n_0 \sim 1$) the latter reference predicts a power law $\sim \lambda^{1/3}$ for the conversion rate which we can exclude on the basis of direct numerics, TWA, and our analytical calculation below. We therefore believe that the actual technical calculation of Ref. [12] is flawed. However, the conceptual basis for the emergence of power laws (as opposed to exponentials such as in the original Landau Zener problem) remains valid: in driven classical dynamical systems, certain invariants of the autonomous limit of the evolution (viz. the action picked up upon traversal of periodic trajectories) become weakly time dependent. While in most regions of phase space this time dependence is very (exponentially in the driving rate) weak, it can become strong (algebraic) in the vicinity of singular

points. Such points reflect the presence of nonlinearities in the Hamiltonian, which in our problem are due to interactions. When the driving parameter sweeps across a singular point, the topology of trajectories in phase space (and thence the action picked up along these trajectories) may change profoundly. It is the dynamics of these processes which we will investigate in the following.

Specifically, we will consider the system in the following limit,

$$N \gg 1, \quad \frac{\lambda}{g^2} \ll 1, \quad \frac{1}{N^\alpha} \ll \frac{\lambda}{g^2} \log N \ll 1, \quad (37)$$

where $\alpha > 0$ is an arbitrary positive number.

We consider the Hamiltonian (4) in the limit of c -number valued operators. Using a number-phase decomposition of the boson field, $b \rightarrow \sqrt{ne^{i\varphi}}$, and a polar representation of spin variables, $S_z = S \cos \theta$, $S_x = S \sin \theta \cos \xi$, $S_y = S \sin \theta \sin \xi$, it assumes the form

$$H(n, \varphi, \theta, \xi) = -\lambda tn + \lambda t \frac{N}{2} \cos \theta + g\sqrt{Nn} \sin(\theta) \cos(\xi - \varphi).$$

The two angles ξ and φ can be combined into one angle $\phi = \xi - \varphi + \pi$ (where π is added for later convenience). The angle θ and the boson number n are not independent, but are related via the conservation law

$$n = \frac{N}{2} (1 - \cos \theta). \quad (38)$$

This allows to trade θ for n . It is then convenient to rescale

$$n \rightarrow Nn, \quad (39)$$

so that the new n varies from 0 to 1, with the last value taken when all particles are bosons. Similarly, the rescaled initial condition reads

$$n_0 \sim \frac{1}{N}.$$

It is also advantageous to replace

$$H \rightarrow NH. \quad (40)$$

Throughout this section, we will work with the rescaled variables as described here, restoring the original variables at the end.

Finally, without any loss of generality we can set $g = 1$ since it can be scaled out by the appropriate rescaling of t . We arrive at the Hamiltonian

$$H = -\gamma n - 2n\sqrt{1-n} \cos(\phi), \quad \gamma = 2\lambda t. \quad (41)$$

Here ϕ and n play the role of the coordinate and momentum, canonically conjugate to each other. The equations of motion of these variables read as

$$\begin{aligned} d_t n &= -\partial_\phi H, \\ d_t \phi &= \partial_n H. \end{aligned} \quad (42)$$

These equations of motion need to be supplied with initial conditions. In this section, where the emphasis is on the calculation of the 'typical' number of produced bosons, we will supply the dynamical system (42) with a fixed value of the initial n . We take this to be the typical value $n_0 \sim 1/N$ of the (rescaled) distribution (34). (As we will see below in the slow limit $n(t \rightarrow \infty)$ has only logarithmic sensitivity to n_0 and precise form of the initial value is not important, as long as we are interested in estimating mean conversion rates. For the same reason the initial fluctuations in the angle θ , which we ignored, are also unimportant.)

We begin our analysis of the initial value problem by plotting lines of constant energy $H = \text{const.}$ at different fixed values of γ . Examples are shown in Figs. 4, 5, 6 and 7. The lines shown in these figures represent

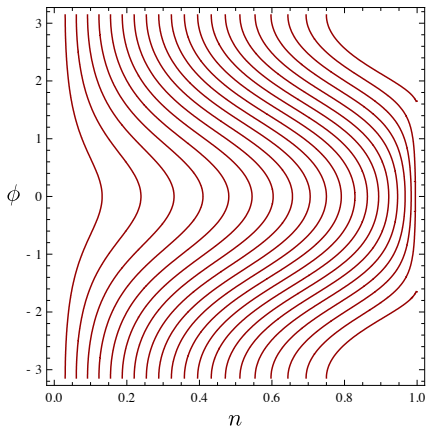


FIG. 4: Trajectories of H at $\gamma = -3$. Here the horizontal axis is n , and the vertical axis is ϕ .

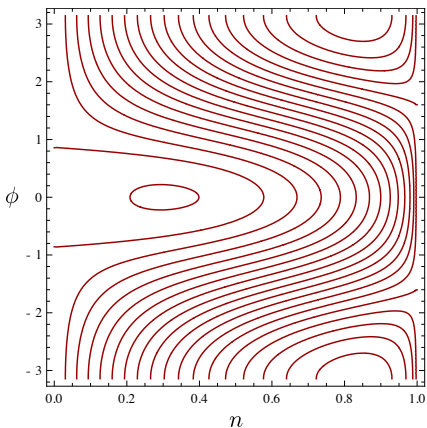


FIG. 5: Trajectories of H at $\gamma = -1.3$

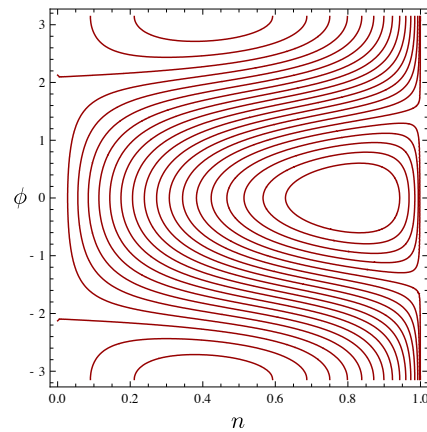


FIG. 6: Trajectories of H at $\gamma = 1$

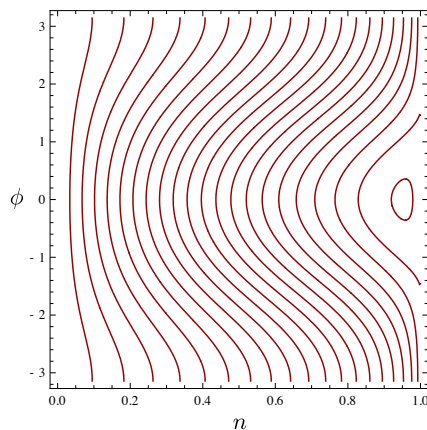


FIG. 7: Trajectories of H at $\gamma = 4$.

trajectories of the Hamiltonian for γ fixed. In our problem, γ changes in time. However, at small λ , γ changes slowly. Thus the system will follow one of the trajectories for some time before it will slowly drift to a different trajectory due to the slowly changing γ .

B. Fixed Points

An important role in our analysis is played by the fixed points of the Hamiltonian system [10, 12]. This is where the trajectory consists of a single point (γ still kept fixed.) They are found by solving

$$\frac{\partial H}{\partial \phi} = 0, \quad \frac{\partial H}{\partial n} = 0. \quad (43)$$

There are two families of solutions (n, ϕ) of these equations. The first exists for $\gamma > -2$ and has $\phi = 0$, $(n, \phi) \equiv (n_1(\gamma), 0)$, where

$$n_1(\gamma) = \frac{12 - \gamma^2 + \gamma\sqrt{12 + \gamma^2}}{18}. \quad (44)$$

This solution appears at $n_1(2) = 0$ at $\gamma = -2$. As γ is increased above -2 , $n_1(\gamma)$ starts increasing until it asymptotically reaches $\lim_{\gamma \rightarrow \infty} n_1(\gamma) = 1$.

The other solution has $\phi = \pi$, and requires $\gamma < 2$. It is given by

$$n_2(\gamma) = \frac{12 - \gamma^2 - \gamma\sqrt{12 + \gamma^2}}{18}. \quad (45)$$

This solution appears at $n_2(2) = 0$. As γ is decreased below 2, $n_2(\gamma)$ starts increasing and asymptotes as $\lim_{\gamma \rightarrow -\infty} n_2(\gamma) = 1$.

We thus arrive at the following picture. Initially the number of bosons is given by $n = n_0 \sim 1/N$, at undetermined ϕ . The coordinates n and ϕ evolve according to the approximate Hamiltonian $H \approx -\gamma n$, where we used that at large negative times γ is very large to keep only the leading order contribution in this scale. This means that the motion is initially given by

$$n = \frac{1}{N}, \quad \phi = -\gamma t. \quad (46)$$

However, as γ is increasing, the fixed point $n_1(\gamma)$, given by (44), appears at the critical value $\gamma = -2$. As γ keeps increasing the point moves towards larger values of n . At a certain moment of time, the trajectory $(n, \phi)(t)$, will drastically change: it will no longer perform extended propagation in ϕ -directions but start winding around the fixed point Eq. (45). Eventually, at some large positive γ , the trajectory will stop winding around the fixed point and merge into a new extended trajectory, which at large positive γ will take the form

$$n = n_b, \quad \phi = -\gamma t. \quad (47)$$

n_b is the final number of bosons we are trying to calculate.

The fixed point $n_1(\gamma)$ has an intimate relationship with the instantaneous ground state of the quantum Hamiltonian Eq. (4). This is discussed in more details in Appendix C.

C. Adiabatic invariants

In the classical mechanics of slowly parameter dependent Hamiltonians there exist approximately conserved ‘adiabatic invariants’ [35]. Generally, adiabatic invariants assume the form of action integrals $\sim \int p dq$, where the integral is over one full revolution of the system’s motion. In the present context, the invariant is given by

$$I = \frac{1}{2\pi} \int d\phi n. \quad (48)$$

To further elucidate the meaning of the adiabatic invariant in our problem, let us go back to the angular variable θ according to Eqs. (38), (39). We find

$$I = \frac{1}{4\pi} \int d\phi (1 - \cos \theta). \quad (49)$$

This is nothing but the area of a part of the surface of a unit radius sphere bounded by the trajectory $\theta(\phi)$, where θ and ϕ are thought of as spherical angles, divided by 4π . This geometric definition of the adiabatic invariant implies an ambiguity which needs to be resolved. A closed trajectory on the surface of a unit sphere separates the surface into two domains, and one needs to pick the area of one of them.

We define the adiabatic invariant as the area (divided by 4π) of the domain on the surface of the unit sphere bounded by the trajectory such that the trajectory, when the domain is viewed from above, encircles the domain in the counter-clockwise direction. We note that this definition does not always conform to the algebraic expression Eq. (48). However, it results in the adiabatic invariant continuous under smooth deformations of the trajectory, while Eq. (48) can be discontinuous under those transformations.

For trajectories such as the initial trajectory Eq. (46) or the final trajectory Eq. (47) the adiabatic invariant is very easy to calculate. Initially the trajectory encloses the north pole of the sphere in the counter-clockwise direction, to result in

$$I_{\text{initial}} \equiv I(\gamma \rightarrow -\infty) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi n(\gamma \rightarrow -\infty) \sim \frac{1}{N}.$$

At the end of the process Eq. (47) implies that the trajectory now encloses the south pole in the counter-clockwise direction. The adiabatic invariant is now given by

$$I_{\text{final}} \equiv I(\gamma \rightarrow \infty) = 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi n(\gamma \rightarrow \infty) = 1 - n_b. \quad (50)$$

In the limit $\lambda \rightarrow 0$, I is conserved.

$$I_{\text{final}} = 1 - n_b \sim \frac{1}{N}. \quad (51)$$

This implies that at the end of the process all particles become bosons, $n_b \approx 1$ (with $1/N$ factor coming from a quantum uncertainty irrelevant in the large N limit).

As the rate λ is increased, the adiabatic invariant starts changing with time. In the next section, we will review the general theory of adiabatic action changes, and then apply it to our particular problem at hand. Readers not interested in details of the adiabatic dynamical evolution of the system may just note the final result of our analysis, Eq. (63).

D. The theory of approximate conservation of the adiabatic invariants

The theory of approximate conservation of adiabatic invariants is a well developed subject, described in detail in Ref. [35]. One of its main results is that the change of an adiabatic invariant during some time dependent process is usually exponentially suppressed at slow rate,

$$\Delta I \sim \exp\left(-\frac{\text{const}}{\lambda}\right), \quad (52)$$

where λ is the rate of the process. In fact, one may interpret the standard Landau-Zener transition probability in terms of this behavior.

However, the theory also states that if a singularity develops at a certain time during the evolution, Eq. (52) breaks down and gets replaced, typically, by a power law. We are going to see that this is indeed the case in our problem.

For the convenience of the reader, we briefly summarize the essentials of the theory of adiabatic invariants, as discussed in chapters 49 and 50 of Ref. [35]. Consider a Hamiltonian $H = H(\phi, n, \gamma)$ depending on a pair of conjugate variables (ϕ, n) , and a slowly varying parameter γ . For starters, assume γ to be constant. Since H describes a system with only one degree of freedom, we have integrability, and it is convenient to transform to canonical, or ‘action-angle’ variables. The action variable is defined as

$$I = \frac{1}{2\pi} \oint d\phi n(E, \phi, \gamma),$$

where the integral is over a closed trajectory of the system’s evolution. Such curves are specified by a value of the conserved energy, E , and, in our case, by the value of γ . We thus have a functional relation $I = I(E, \gamma)$. Now, consider the so-called abbreviated action

$$S(\phi, E, \gamma) \equiv \int_{\phi_0}^{\phi} d\phi' n(E, \phi', \gamma),$$

where the integral extends only over a certain segment of the trajectory. The relation $I = I(E, \gamma)$ may be inverted to express S as

$$S(\phi, E(I, \gamma), \gamma) = S(\phi, I, \gamma)$$

as a function of the coordinate and the action variable. The abbreviated action is the generator of a canonical transformation $(\phi, n) \rightarrow (I, w)$ from the old coordinates to the (conserved) action, and a conjugate angular variable. Specifically, we have the relation $n = \partial_{\phi} S(\phi, I, \gamma)|_I$ and define the angular variable as $w \equiv \partial_I S(\phi, I, \gamma)|_{\phi}$. The equations of motion in the new variables read

$$\begin{aligned} d_t I &= -\frac{\partial H'}{\partial w}, \\ d_t w &= \frac{\partial H'}{\partial I}, \end{aligned}$$

where $H' = H'(I, w, \gamma)$ is the Hamiltonian expressed in terms of new variables. For an autonomous system ($\gamma = \text{const.}$), $H'(I, w, \gamma) = H(I, \gamma) \equiv E(I, \gamma)$ is but the old Hamiltonian expressed in terms of the action variable. In this case, the action is conserved, and the angle varies as $w = t \partial_I E$. During each period of the motion, the action changes by an amount $\Delta S = 2\pi I$. The relation $w = \partial_I S$ then implies a change of w by 2π . Thus,

$$w(t) = t \frac{2\pi}{T}.$$

For a non-autonomous system, classical mechanics states that

$$H'(I, w, \gamma) = H(I, \gamma) + \frac{\partial S}{\partial t} = H(I, \gamma) + \frac{\partial S}{\partial \gamma} d_t \gamma,$$

where S on the r.h.s. must be expressed as a function of (I, w) after the parameter differentiation. In other words,

$$H'(I, w, \gamma) = H(I, \gamma) + \Lambda(I, w, \gamma) d_t \gamma,$$

where

$$\Lambda(I, w, \gamma) = \left. \frac{\partial S(I, \phi, \gamma)}{\partial \gamma} \right|_{\phi=\phi(I, w, \gamma)}. \quad (53)$$

This then means, that the action changes, and it does so according to

$$d_t I = -\frac{\partial \Lambda(I, w, \gamma)}{\partial w} d_t \gamma. \quad (54)$$

The change of the action over a finite interval of time then follows as

$$\Delta I = - \int dt \frac{\partial \Lambda(I, w, \gamma)}{\partial w} d_t \gamma = -2\lambda \int \frac{dw}{\omega} \frac{\partial \Lambda(I, w, \gamma)}{\partial w}, \quad (55)$$

where

$$\omega \equiv \frac{2\pi}{T}$$

is the characteristic frequency of the trajectory. For time independent γ , the integral is over one full interval of a w -periodic integrand and vanishes; it is temporal variations in γ that give it its finite value. The relations above express the evolution of the action entirely in terms of quantities that do not contain explicit time dependence. One may argue that ΔI evaluate to Eq. (52), unless the time dependence of ω^{-1} and R contains singularities. This will turn out to be the case in our problem.

Figs. 4, 5, 6 and 7 show examples of trajectories that enter the computation of action integrals for our dynamical system. (Each of these trajectories corresponds to a particular value of I while w parameterizes the trajectory.) In computing the above integrals, we will then be met with the following scenario: consider a trajectory, close to the left boundary of phase space and at an initial value $\gamma \ll 0$. Initially, that trajectory will be ‘open’, i.e. its angular variable will periodically run from 0 to 2π , at moderately changing n . To the trajectory we may assign an instantaneous value of energy $H(n, \phi, \gamma)$, which changes on account of increasing γ . As γ approaches the value -2 from below, we run into a singularity: at a certain value $\gamma = -2 + \epsilon_0$, $0 < \epsilon_0 \ll 1$, the energy vanishes, $H(n, \phi, -2 + \epsilon_0) = 0$. That point marks a drastic change in the topology of the trajectory, it changes from open to closed. See Fig. 5 for an example of a closed trajectory, and a ‘critical’ trajectory with $H = 0$. The latter begins and ends at $n = 0$, and has the form of a horizontally

aligned horseshoe. The passage time through the critical trajectory is infinite, and it is logarithmically divergent on the closed trajectories for $\epsilon \searrow \epsilon_0$ (in contrast, it can be checked that another horseshoe-shaped trajectory, which starts and ends at $n = 1$ and can be seen on Fig. 7, is not critical and its passage takes finite amount of time). The action integral receives its dominant contribution from a range of ϵ -values above ϵ_0 .

To describe this situation algebraically, we expand the Hamiltonian Eq. (41) for small ϵ , n and ϕ , where

$$\gamma = -2 + \epsilon$$

as

$$H = n(\phi^2 - \epsilon + n).$$

The singular trajectory is given by $H = 0$, or $n = \epsilon - \phi^2$. Its frequency is zero (the period is infinity) and its adiabatic invariant is given by

$$I = \frac{1}{2\pi} \int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} d\phi (\epsilon - \phi^2) = \frac{2}{3\pi} \epsilon^{3/2}. \quad (56)$$

The trajectories inside the singular trajectory have $H < 0$ (one of these is shown in Fig. 5). On these closed trajectories, n is a two valued function of ϕ , where the two branches $n_{\pm}(\phi) = \frac{1}{2} \left(\epsilon - \phi^2 \pm \left((\phi^2 - \epsilon)^2 + 4H \right)^{1/2} \right)$ represent the right and left bending arc as ϕ varies in $-\sqrt{\epsilon - 2\sqrt{-H}} \leq \phi \leq \sqrt{\epsilon - 2\sqrt{-H}}$. The action of these curves equals the area enclosed by the trajectories, divided by 2π , or

$$I = \frac{1}{2\pi} \int_{-\sqrt{\epsilon - 2\sqrt{-H}}}^{\sqrt{\epsilon - 2\sqrt{-H}}} d\phi \sqrt{(\phi^2 - \epsilon)^2 + 4H} \approx \frac{\epsilon^{3/2}}{2\pi} \left(\frac{4}{3} - \frac{H}{\epsilon^2} \left(\log \left[-\frac{H}{16\epsilon^2} \right] - 1 \right) \right). \quad (57)$$

This expression is approximate, valid for small $|H| \ll \epsilon^2$. The corresponding frequencies are given by

$$\omega = \frac{\partial H}{\partial I} = -\frac{2\pi\sqrt{\epsilon}}{\log \left(-\frac{H}{16\epsilon^2} \right)}. \quad (58)$$

These frequencies indeed vanish logarithmically as $H \nearrow 0$ approaches zero, confirming that the $H = 0$ trajectory is critical.

The mechanical system we need to study starts its evolution at $t \rightarrow -\infty$ by moving according to Eq. (46). Subsequently, at a time where γ is slightly larger than -2 , the system crosses the singular trajectory and starts moving around the fixed point Eq. (45). It is at this time that the adiabatic invariant receives most of its increase.

We estimate the increase by first calculating S for the part of the trajectory represented by $n_-(\phi)$ where $\phi < 0$

$$S = \frac{1}{2} \int_0^\phi d\phi' \left(\epsilon - \phi'^2 - \sqrt{(\phi'^2 - \epsilon)^2 + 4H} \right).$$

In this formula, $H = H(I)$ must be understood as a function of I , which can be found by inversion of (57). We then calculate Λ according to (53):

$$\Lambda = \frac{1}{2} \int_0^\phi d\phi' \left[1 - \frac{\epsilon - \phi'^2 + 2\frac{\partial H}{\partial \epsilon}}{\sqrt{(\epsilon - \phi'^2)^2 + 4H}} \right].$$

where ϕ has to be substituted by $\phi = \phi(I, w, \lambda)$ after the differentiation. We now use this result to compute the time variation of the action according to (54). Using that the w -dependence of Λ is in $\phi(w, I, \lambda)$, and that $d_t \gamma = 2\lambda$, we have

$$d_t I = -2\lambda \frac{\partial \Lambda}{\partial \phi} \frac{\partial \phi}{\partial w} = -\lambda \left[1 - \frac{\epsilon - \phi^2 + 2\frac{\partial H}{\partial \epsilon}}{\sqrt{(\epsilon - \phi^2)^2 + 4H}} \right] \frac{\partial \phi}{\partial w}.$$

Eventually, this quantity has to be integrated over time, and to do so, we need information on the time dependence of $\phi(I, w, \gamma) = \phi(I, \omega t, \gamma)$. The equations of motion tell us

$$d_t \phi = \partial_n H = -\sqrt{(\epsilon - \phi^2)^2 + 4H}. \quad (59)$$

We next assume that during the time when the invariant I receives most of its increase, ϕ takes values such that the r.h.s. of this equation vanishes. The validity of this presumption will be checked in the end of the section. This implies

$$\phi \sim -\sqrt{\epsilon}. \quad (60)$$

Indeed, everything done so far implies $|H| \ll \epsilon^2$ and so the right hand of Eq. (59) approximately vanishes if Eq. (60) holds.

Using this stationarity condition, and $\partial_w \phi = \omega^{-1} d_t \phi$ we obtain

$$d_t I \simeq \frac{2\lambda}{\omega} \left(\sqrt{-H} - \partial_\epsilon H \right).$$

In this equation $H = H(I, \epsilon) \ll \epsilon^2$ is implicitly defined by (57). This gives

$$\frac{\partial H}{\partial \epsilon} = -\frac{\partial I}{\partial \epsilon} \left(\frac{\partial I}{\partial H} \right)^{-1} \simeq \frac{2\epsilon}{\ln \left(-\frac{H}{16\epsilon^2} \right)},$$

a term that is larger than the $\sqrt{-H}$ contribution to $d_t I$. Using (58), we arrive at the estimate

$$d_t I \simeq \frac{2\lambda\epsilon^{1/2}}{\pi}.$$

Using that $\epsilon = 2 + 2\lambda t$, we integrate this expression over ϵ from $\epsilon = \epsilon_0$, remembering that $I = 2\epsilon_0^{3/2}/(3\pi)$ when $\epsilon = \epsilon_0$, to find

$$I \simeq \frac{2}{3\pi} \epsilon^{3/2}. \quad (61)$$

This expression looks formally identical to Eq. (56), but has a different meaning. While Eq. (56) is the adiabatic invariant of the critical trajectory, Eq. (61) signifies that this expression remains approximately true even at later times when the system no longer follows the critical trajectory.

We finally need to find the maximum value $\epsilon \equiv \epsilon^*$ for which Eq. (61) still holds (at even higher values of ϵ , the adiabatic invariant will stop growing and just oscillate about its average value). The criterion we will use to determine that value reads $w = \pi$, i.e. we demand that the system proceeded along half a period of the critical trajectory. At larger instances of time (ϵ) the transit into the domain of oscillatory motion and no further systematic action increase has taken place. In practice, the fixation of ϵ^* turns out to be somewhat tedious, and we have relegated it into appendix D. As a result we obtain

$$\epsilon^* \sim \left(\frac{3}{2} \lambda \log N \right)^{\frac{2}{3}}. \quad (62)$$

Finally, with the help of Eq. (51) this leads to our final answer

$$n_b \simeq N \left(1 - \frac{\lambda}{\pi g^2} \log N \right), \quad (63)$$

where we have re-introduced the coupling strength g and returned to the original (unrescaled) n_b . Note that in the derivation above we relied only on the fact that the unrescaled $n_0 \sim 1$ due to quantum fluctuations. If we start from the state where n_0 is non-zero (but smaller than N) due to initial occupancy of the molecular state the result (63) remains valid with the only difference that the argument of the logarithm should be changed to N/n_0 . This is qualitatively different from the results of Ref. [12] where an instant crossover to a $\sim \lambda^{1/3}$ power law the moment $n_0 \gtrsim 1/N$ was predicted.

VII. TWA AND COMPARISON WITH NUMERICAL SIMULATIONS

In this section, we will apply a combination of numerical diagonalization and the TWA to obtain accurate results for both, the mean value of n_b and its distribution. For moderate values of N , the Schrödinger equations for Eq. (4) can be solved directly. Indeed, the Hilbert space corresponding to that Hamiltonian, in the sector $S_z + b^\dagger b = N/2$, has only $N + 1$ states. Thus the Hamiltonian reduces to a $(N + 1)$ -dimensional matrix. This matrix can be diagonalized at reasonable numerical cost up to values $N \lesssim 10^3$ (see Appendix C for an example of how this procedure can be set up).

At larger values of N , we simulate the classical equations of motion, with initial drawn from the quantum Wigner distribution Eq. (34) – the TWA. In this way, we

may obtain results for significantly larger values of N . In the following we will apply both methods interchangeably. However, before doing so, let us first test the accuracy of TWA by comparison to numerical diagonalization at values of N where both methods are applicable.

A. TWA vs direct diagonalization

In section V we have argued that the exactness of the TWA in the linear regime entails its applicability to the whole range of driving parameters. To back up this claim, let us compare TWA results to those obtained by numerical diagonalization. In Fig. 8 we show $\langle n_b \rangle(\lambda)$ for two values of $N = 2S$: $N = 64$ and $N = 128$. The solid and dashed lines represent the exact and TWA solutions, respectively. There is no visible difference between them. To demonstrate that TWA is not actually exact we show in the inset the difference between TWA and numerical results multiplied by a factor of 100. Clearly for $N = 128$ the accuracy of TWA is better which signals that as N increases TWA gives results of increasing precision.

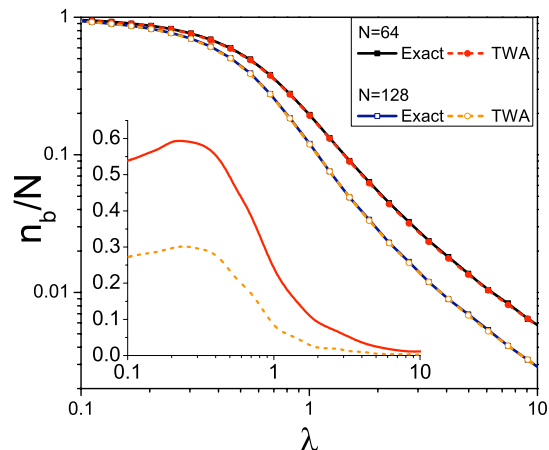


FIG. 8: Dependence of the scaled number of bosons n_b/N as a function of the rate λ . The two pairs of curves correspond to $N \equiv 2S = 64$ and $N = 128$. The solid lines represent the exact solution while the dashed lines are the semiclassical TWA. The inset shows the difference between exact and semiclassical results ($N = 64$ red solid line and $N = 128$ orange dashed line) multiplied by a factor 100.

In Fig. 9 we plot the relative number fluctuations $\delta n_b/N = \sqrt{\langle n_b^2 \rangle - \langle n_b \rangle^2}/N$ as a function of λ for $N = 64$. As with the dependence $n_b(\lambda)$ the difference between TWA and exact results is minuscule and it vanishes fast as N gets larger. These results make us confident that TWA represents a high precision method which becomes asymptotically exact in the limit $N \rightarrow \infty$.

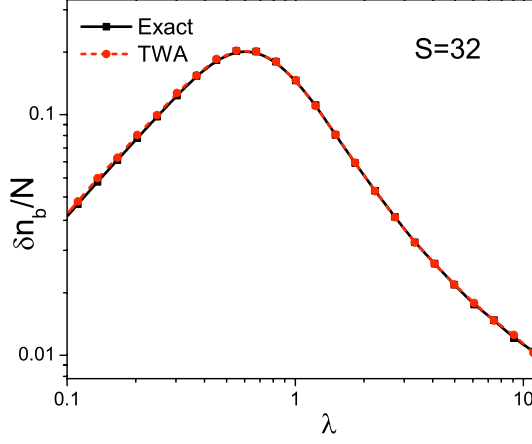


FIG. 9: Dependence of the scaled variance of the number of bosons $\delta n_b/N$ after the process as a function of the rate λ for $N = 64$. The two curves represent the exact and the TWA results.

B. Combined application of TWA and direct diagonalization

In the following we will analyse our system by combined application of the two numerical methods. We will not indicate which method was used for which particular curve, unless necessary. In all numerical simulations we set $g = 1$. (This can be always achieved by the rescaling $\lambda \rightarrow \lambda g^2$.) Unless stated differently, the notation $n_b(\lambda)$ refers to the mean value of the number of bosons.

In Fig. 10 we plot $n_b(\lambda)$ for different values of N . For small N the function $n_b(\lambda)$ is qualitatively similar to an exponential form: $n_b(\lambda)/N \propto 1 - 1/x = 1 - \exp(-\pi/\lambda)$, as with the conventional two-level Landau-Zener problem. However, as N increases this exponential behavior disappears and the dependence becomes much closer to the linear one predicted in Eq. (63). To demonstrate the approach to linear scaling, we plot $n_b(\lambda)$ for large values of N up to $N \sim 10^8$ in Fig. 11. The dashed line is the fit to the linear dependence: $n_b/N \approx 1 - 0.28\lambda \ln N$. Note that the prefactor 0.28 is slightly different from the $1/\pi$ in Eq. (63). But given that the analytic result was obtained using a series of approximations we find this level of agreement to be satisfying.

Fig. 12 shows the results of numerical simulations at $N = 10^3$, $N = 10^4$, $N = 10^6$ and $N = 10^8$, plotted as $(1 - n_b/N)/\log N$ vs $\lambda/(\pi g^2)$. The straight line in the plot represents the deep adiabatic limit Eq. (63), while

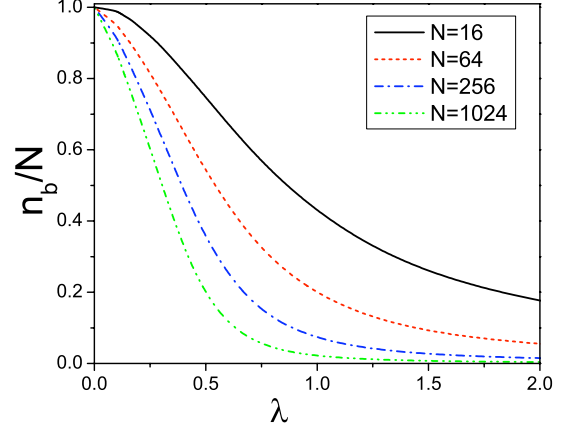


FIG. 10: Number of remaining bosons n_b vs the rate λ for different values of N . It is clear that as N increases the dependence goes from exponential form characteristic for Landau-Zener process to the linear one predicted analytically (see Eq. (63))

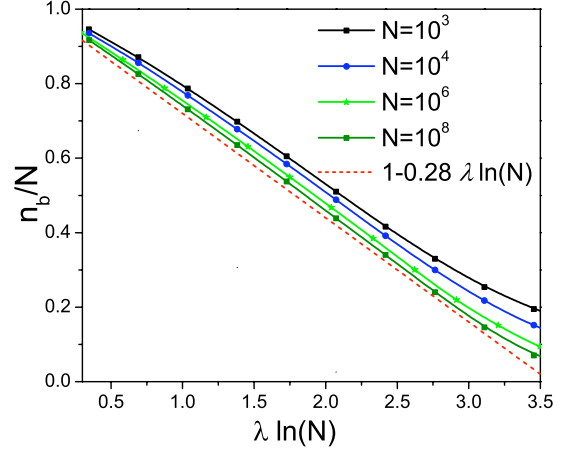


FIG. 11: Same as in Fig. (10) but for large values of N . Note that the horizontal axis is $\lambda \ln N$.

the second continuous line is the solution of the kinetic equation Eq. (31) at $N = 10^3$ only.

We see that the deep adiabatic relation Eq. (63) is confirmed in the limit of slow rates. As to the solution of the kinetic equation, while it describes well the relatively fast rates, it deviates from the numerics at slower rates and eventually breaks down when it saturates at nonzero value when $\lambda \rightarrow 0$.

Next, let us discuss the distribution function $P(n_b)$. In the fast limit we have a well defined analytic prediction

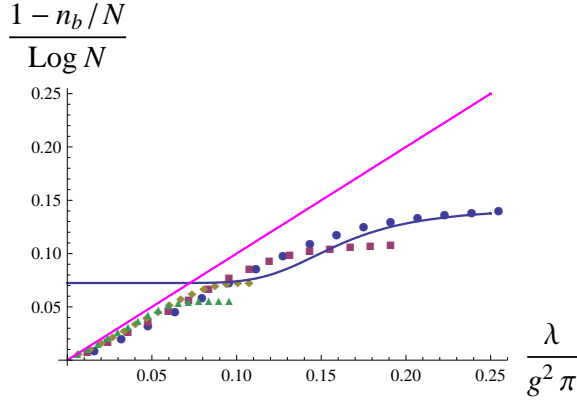


FIG. 12: Number of remaining bosons n_b vs the rate λ for different values of N . Here circles represent $N = 10^3$, squares $N = 10^4$, rhombi $N = 10^6$, and triangles $N = 10^8$. The continuous lines represent our analytic solutions, as discussed in the text.

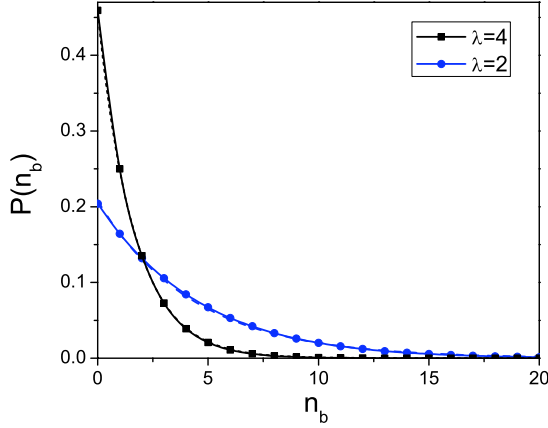


FIG. 13: Distribution function of n_b for $N = 128$ and two values of the rate: $\lambda = 2$ and $\lambda = 4$. The distributions are discrete and the lines are the guide to the eye. The dashed lines, which are barely distinguishable from the numerically computed solid lines, are the analytic prediction (14.)

given by Eq. (14) While for the slow limit we do not have an analytic prediction for the distribution function, one can make a good ansatz based on the result Eq. (63). Note that this result was obtained for a particular choice of $n_0 = 1$. More generally for each fixed initial value of n_0 it should read:

$$n_b(n_0)/N \approx 1 - \frac{\lambda}{C_1} \ln(C_2 N/n_0), \quad (64)$$

where C_1 is a constant close to π and C_2 is another constant, which in general can depend on λ . Finding this constant is beyond accuracy of our analytic derivation. Because we know the distribution of n_0 :

$P(n_0) = 2 \exp[-2n_0]$ we can invert Eq. (64) and derive the distribution function of n_b :

$$P(n_b) \approx 2 \frac{C_1}{\lambda} C_2 e^{-\rho} \exp[-2C_2 N e^{-\rho}], \quad (65)$$

$$\rho = \frac{C_1}{\lambda} \left(1 - \frac{n_b}{N}\right).$$

The function $P(n_b)$ above describes the Gumbel distribution, which often appears in the context of the extreme value statistics.

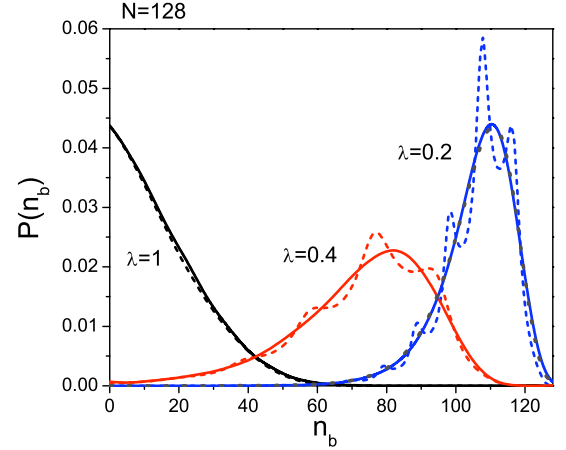


FIG. 14: Distribution function of n_b for $N = 128$ at $\lambda = 1, 0.4, 0.2$. The solid lines are the result of the semiclassical TWA approximation and the dashed lines are the distributions obtained by exact diagonalization. The dotted grey line (which almost coincides with the solid blue line) is the fit to the Gumbel distribution (65) with $C_1 \approx 3.05$ and $C_2 \approx 0.03$.)

We plot the distribution function $P(n_b)$ in Fig. 14. The solid lines are the result of the TWA approximation and the dashed lines are the exact result. At slow rates the distribution function becomes highly asymmetric. At $\lambda = 0.2$ it is very well fitted by the Gumbel distribution (65) (grey dotted line) with $C_1 \approx 3.05$ (instead of π) and $C_2 \approx 0.03$. The fact that C_2 is so small probably indicates that this constant depends on λ . We checked that the fit also works well for $N = 256$ with the same constants C_1 and C_2 . From Fig 14 it is apparent that the semiclassical approximation correctly predicts the Gumbel-like shape of the distribution function at small λ but misses the oscillations on top of this shape. These oscillations have a minor effect on the expectation value of n_b and on its fluctuations. However, they are not negligible and they do not vanish if we increase N (at least they persist up to $N = 512$). We are not sure what the origin of these oscillations is. It is interesting to point out that a similar interpolation of the distribution

function from the exponential to the Gumbel form was found in the completely different context of interference between two independent Luttinger liquids [36].

C. Discussion

The result (63) implies that the thermodynamic limit of our system never behaves adiabatically: no matter how small the driving, in the limit $N \rightarrow \infty$, the adiabatic limit becomes elusive. Such type of behavior has been found earlier [37] for more general low-dimensional bosonic systems near instabilities such as second order phase transitions. The giant quantum fluctuations accompanying this non-adiabatic behavior result from the inversion of the classical ground states of the participating systems. Zero point fluctuations are required to trigger a macroscopic inversion of state occupancies and the stochastic nature of this initiation process generates vast fluctuations at later stages.

Finally, it is worthwhile putting the results above in a more general context: a number of previous works [38, 39, 40, 41, 42, 43, 44] have addressed the slow dynamics of low-dimensional *critical* systems. In all these cases it has been argued that the number of excitations produced upon slow driving scales as a power law of the rate (as opposed to the exponential dependence Eq. (2) for the Landau-Zener problem.) The exponent of this power law is related to the critical exponents characterizing the phase transition [38, 39] under consideration. In low dimensions with their relatively higher density of low energy states there is a stronger tendency for smaller powers, i.e. less adiabatic regimes. In a sense, our 'infinite-range interaction' or 'mean field' system exemplifies these structures in the extreme setting of a zero-dimensional system.

VIII. APPLICATION TO OTHER DRIVING PROCESSES

We spent most of this paper discussing driving processes where the bosonic sector of the system was initially empty (or there were no atoms in the language of the atom-molecular conversion systems Eqs. (8) and (10).)

For these initial conditions, quantum fluctuations are needed to jump start boson (or molecule) production, and this is what ultimately leads to the power law (63). We here briefly discuss the 'inverse' problem, where initially all N particles were bosons (or atoms). This variant was previously discussed in Ref. [27]. What makes it different, is that no quantum fluctuation are required to start the conversion process. Solution of the classical equations of motion Eq. (41) at inverted driving rate, $\gamma = -2\lambda t$, and with initial condition $n(t \rightarrow -\infty) = N$ captures the dynamics of the process and quantum fluctuation introduce only minor corrections.

As a result of a straightforward adaption of the discussion of section IV, we find that the number of bosons n_b for Eq. (4) in the infinite future behaves as

$$n_b \simeq \frac{N}{2 - e^{-\frac{\pi g^2}{\lambda}}}. \quad (66)$$

At fast rates, this is equivalent to $n_b \simeq N e^{-\frac{\pi g^2}{\lambda}}$, $\lambda \gg \pi g^2$, which is but the standard Landau-Zener prediction. As the rate slows down, below $x = e^{\pi g^2/\lambda} \sim 1$ the Landau-Zener prediction breaks down. (Notice that the initial [linear] regime in the direct problem was much more robust.)

Taken at a face value, Eq. (66) predicts that at slow driving rates n_b saturates at $1/2$. This cannot be correct since Eq. (66) must break down at slower rates. A detailed analysis involving the quasiclassical Hamiltonian Eq. (41) with the initial condition $n_0 = 1$, gives

$$n_b/N \simeq 0.057 \lambda, \quad (67)$$

where the constant of proportionality 0.057 was determined numerically from the linear fit (note that unlike that in Eq. (63), it is N -independent). In Fig. 15 we

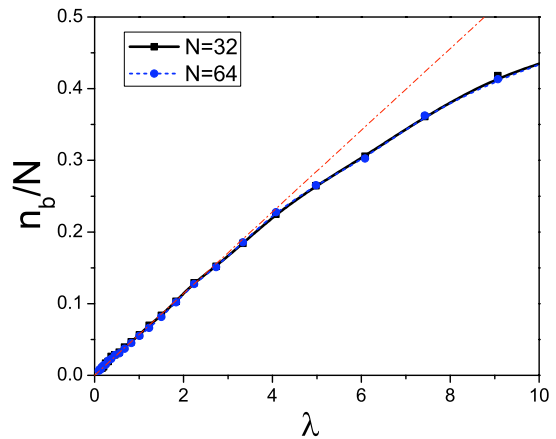


FIG. 15: The number of remaining molecules vs. λ for the inverse problem, where we start from the full molecular condensate and no atoms. Note that the results for $N = 32$ and $N = 64$ almost identically coincide. The thin red line represents the linear fit at slow rates: $n_b/N \approx 0.057\lambda$.

show numerical data for the dependence $n_b(\lambda)$ for the inverse problem for two values of $N = 32$ and $N = 64$. The thin red line represents the linear fit at slow rates: $n_b/N \approx 0.057\lambda$, and the numerical data approaches this asymptotic in an N -independent manner.

Thus the number of remaining bosons, a measure of deviation of adiabaticity, remains proportional to the rate at slow rates, as opposed to the exponential suppression

of the standard Landau-Zener process. Even though one approaches the adiabatic regime when the Landau-Zener parameter is close to 1, the deviation from adiabaticity is still significant and much larger than in the standard problem. This may have significant implications for the experiments such as those reported in Refs. [20, 21], provided the initial state is chosen to be atoms and one is interested in the molecule production. We also note that, as far as we can see, this point was not discussed in Ref. [27].

In some applications it may be important [16] to explore the sector of the theory where the total number of particles is much smaller than N (i.e. much smaller than the number of two-level systems in Eq. (6), or smaller than the number of fermionic pair levels in Eq. (7).) Once more, it is easiest to analyze the system in its spin- $N/2$ incarnation Eq. (4). Translated to the language of this Hamiltonian a sparse initial occupation of fermionic Hilbert space means that we start the process at zero bosons and a spin that is pointing nearly downwards (while the total spin continues to define the dimension of state space, $S = N/2$.) The particle conservation law then reads $S_z + n = -N/2 + \epsilon$, where $\epsilon \ll N$ and $0 \leq n \leq \epsilon$ is the number of bosons. This regime can be modeled in terms of Holstein-Primakoff bosons. However, as the spin continues to move downwards, the regime of Holstein-Primakoff linearizability will not be left. Introducing

$$\begin{aligned} S^+ &= c^\dagger(2S - c^\dagger c)^{1/2}, \\ S^- &= (2S - c^\dagger c)^{1/2}c, \\ S^z &= -S + c^\dagger c, \end{aligned} \quad (68)$$

where $S = N/2$ (the version of Eq. (11) for the spin pointing close to downwards), we find

$$H = \lambda t (b^\dagger b - c^\dagger c) + g (b^\dagger c + c^\dagger b).$$

Here $b^\dagger b + c^\dagger c = \epsilon$. This problem is equivalent to the standard Landau-Zener problem. This proves that in this sector the model reduces to the standard problem and does not have any novel features.

IX. CONCLUSIONS

Let us summarize the findings of our paper. The model described by Eq. (4), (as well as its other incarnations, given by Eqs. (6), (7), and (8) together with (10)), with the initial conditions

$$\lim_{t \rightarrow -\infty} \langle b^\dagger(t)b(t) \rangle = 0,$$

results in a total final number of bosons n_b , defined by

$$n_b = \lim_{t \rightarrow +\infty} \langle b^\dagger(t)b(t) \rangle$$

given by the following expressions:

- For small values of the LZ-parameter,

$$e^{\frac{\pi g^2}{\lambda}} \ll N : \quad n_b = e^{\frac{\pi g^2}{\lambda}} - 1.$$

This result can be obtained within the Holstein-Primakoff method, or by the large- N approximation within the Keldysh formalism.

- At intermediate values,

$$e^{\frac{\pi g^2}{\lambda}} \simeq N : \quad n_b \simeq \frac{N \left(e^{\frac{\pi g^2}{\lambda}} - 1 \right)}{2e^{\frac{\pi g^2}{\lambda}} + N},$$

This can be found using the kinetic equation approach within the Keldysh formalism.

- And at large values,

$$e^{\frac{\pi g^2}{\lambda}} \gg N : \quad n_b \simeq N \left(1 - \frac{\lambda}{\pi g^2} \log(N) \right),$$

This result was found using the quasiclassical approximation together with the adiabatic invariants formalism.

We observe main distinctions between our results and what would have happened had we extrapolated the standard Landau-Zener to our problem (which would have given $n_b = N(1 - e^{-\frac{\pi g^2}{\lambda}})$). First, the transition between the fast rate to slow rate regime happens at

$$\lambda \simeq \frac{\pi g^2}{\log N},$$

the rate which becomes progressively smaller as N is increased (unlike the Landau-Zener result $\lambda \simeq \pi g^2$). Second, if $\lambda \ll \pi g^2 / \log N$, then n_b approaches N as a linear function of λ , much more slowly than the Landau-Zener result which would predict an exponentially fast approach. Finally, particle occupancies in our system show massive quantum fluctuations, comparable in value to the mean particle numbers.

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APPENDIX A: DERIVATION OF EQS. (13) AND (14)

Consider the Heisenberg equations of motion $i\partial_t x = [\hat{H}, x]$, $x = c, b$ corresponding to the Hamiltonian (12),

$$\begin{aligned} i\partial_t b + \lambda t b + g c^\dagger &= 0, \\ -i\partial_t c^\dagger + \lambda t c^\dagger + g b &= 0. \end{aligned} \quad (A1)$$

The structure of these equations suggests the linear combination

$$\begin{aligned} c &= \mu c_0 + \nu b_0^\dagger \\ b &= \rho b_0 + \sigma c_0^\dagger, \end{aligned} \quad (\text{A2})$$

where $c_0 \equiv c(t_0)$, $t_0 \ll 0$ is some time in the distant past when the dynamical evolution is started, and μ, \dots, σ are complex valued functions depending on time. Differentiating the first of the two equations in (A1) once more w.r.t. time and substituting the second equation, we obtain

$$[\partial_t^2 - i\lambda + (\lambda t)^2 - g^2] x = 0, \quad x = c, b. \quad (\text{A3})$$

These two equations are coupled by the initial conditions, which follow from the first order ancestor equations (A1). To make these dependencies explicit, we substitute the expansion (A2) and obtain

$$\begin{aligned} [\partial_t^2 - i\lambda + (\lambda t)^2 - g^2] \kappa &= 0, \quad \kappa = \mu, \nu, \rho, \sigma, \\ \kappa = \mu, \rho : \quad \kappa(t_0) &= 1, \quad \partial_t \kappa(t_0) = i\lambda t_0, \\ \kappa = \nu, \sigma : \quad \kappa(t_0) &= 0, \quad \partial_t \kappa(t_0) = ig. \end{aligned} \quad (\text{A4})$$

Notice that

$$\begin{aligned} \mu &= \rho \text{ and } \nu = \sigma, \\ |\mu|^2 - |\nu|^2 &= 1, \end{aligned} \quad (\text{A5})$$

where the first line is a consequence of (μ, ρ) and (ν, σ) obeying differential equations with identical initial conditions, and the second is enforced by the fact that the time dependent operators $(c, c^\dagger, b, b^\dagger)$ obey canonical commutation relations.

These equations afford a solution in terms of parabolic cylinder functions [8]. All we need to know about these functions is that

$$|\mu|^2 \xrightarrow{t \rightarrow \infty} e^{\pi g^2 / \lambda}. \quad (\text{A6})$$

Using $n_b = \langle c^\dagger c \rangle = \langle (\bar{\mu} c_0^\dagger + \bar{\nu} b_0)(\mu c_0 + \nu b_0^\dagger) \rangle = |\nu|^2$ and Eqs. (A5), (A6) we then obtain Eq. (13).

The actual distribution of the particle number can be obtained by a bit of linear algebra. We begin by condensing Eqs. (A2) into the matrix equation

$$\begin{pmatrix} c \\ b^\dagger \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ \bar{\nu} & \bar{\mu} \end{pmatrix} \begin{pmatrix} c_0 \\ b_0^\dagger \end{pmatrix} \Rightarrow \begin{pmatrix} c_0 \\ b_0^\dagger \end{pmatrix} = \begin{pmatrix} \bar{\mu} & -\nu \\ -\bar{\nu} & \mu \end{pmatrix} \begin{pmatrix} c \\ b^\dagger \end{pmatrix}, \quad (\text{A7})$$

where we used Eqs. (A5). Denoting the occupation number eigenstates of the time dependent operators by $|k, l\rangle$, i.e.

$$\begin{aligned} c|k, l\rangle &= \sqrt{k}|k-1, l\rangle, & c^\dagger|k, l\rangle &= \sqrt{k+1}|k+1, l\rangle, \\ b|k, l\rangle &= \sqrt{l}|k, l-1\rangle, & b^\dagger|k, l\rangle &= \sqrt{l+1}|k, l+1\rangle, \end{aligned}$$

We now expand the vacuum state (of the operators c_0, b_0), $|0\rangle$, in the basis $\{|k, l\rangle\}$:

$$|0\rangle = \sum_{k,l} c_{k,l} |k, l\rangle.$$

To determine the coefficients $c_{k,l}$, we use the defining relation $0 \stackrel{!}{=} c_0|0\rangle \stackrel{(\text{A7})}{=} (\bar{\mu}c - \nu b^\dagger) \sum_{k,l} c_{k,l} |k, l\rangle$. This equation, and its partner $b_0|0\rangle = 0$ generate the set of recursion relations

$$\begin{aligned} c_0|0\rangle = 0 : \quad \mu\sqrt{k+1}c_{k+1,l+1} &= \bar{\nu}\sqrt{l+1}c_{k,l}, \quad k, l \geq 0, \\ \mu c_{k,0} &= 0, \quad k \geq 1, \end{aligned}$$

$$\begin{aligned} b_0|0\rangle = 0 : \quad \bar{\nu}\sqrt{k+1}c_{k,l} &= \mu\sqrt{l+1}c_{k+1,l+1}, \quad k, l \geq 0, \\ \mu c_{0,l} &= 0, \quad l \geq 1. \end{aligned}$$

From these equations we deduce $c_{k,0} = 0$ for $k \geq 0$ and $c_{0,l} = 0$ for $l \geq 0$. The first and third equation then imply $c_{k,l} = 0$ for $k \neq l$. The diagonal terms successively descend from $c_{0,0}$ according to

$$c_{k,l} = \delta_{k,l} c_{0,0} \left(\frac{\bar{\nu}}{\mu} \right)^k.$$

The normalization condition $1 = \langle 0|0\rangle = \sum_{k,l} |c_{k,l}|^2$ generates the additional condition

$$1 = \frac{|c_{0,0}|^2}{1 - \left| \frac{\bar{\nu}}{\mu} \right|^2} = |c_{0,0}|^2 |\mu|^2.$$

We thus arrive at the expansion

$$|0\rangle = \frac{1}{|\mu|} \sum_{k=0}^{\infty} \left(\frac{\bar{\nu}}{\mu} \right)^k |k, k\rangle.$$

From this, the moments of the particle number operator are straightforwardly obtained as

$$c_m \equiv \langle 0|(c^\dagger c)^m|0\rangle = \frac{1}{|\mu|^2} \sum_{k=0}^{\infty} \frac{|\nu|^{2k}}{|\mu|^{2k}} k^m.$$

Comparison with the formulation of moments in terms of the discrete probability distribution $P(n)$

$$c_m = \sum_n P(n) n^m$$

leads to the identification $P(n) = \frac{1}{|\mu|^2} \frac{|\nu|^{2n}}{|\mu|^{2n}}$ or, using Eq. (A5),

$$P(n) = \frac{1}{|\mu|^2} \left(1 - \frac{1}{|\mu|^2} \right)^n. \quad (\text{A8})$$

Substitution of Eq. (A6) then leads to Eq. (14). From this distribution, the mean value is readily obtained as (13).

APPENDIX B: DERIVATION OF THE TRUNCATED WIGNER APPROXIMATION FOR THE SPIN-BOSON MODEL.

1. Truncated Wigner approximation

In this appendix, we first review the TWA for a general system of bosons, and then adapt to our specific spin-boson problem. Consider a (generally time dependent) Hamiltonian formulated in terms of normal ordered bosonic creation and annihilation operators $\mathcal{H}(\hat{\psi}^\dagger, \hat{\psi}, t)$. The hat-notation $\hat{\psi}$ is used to distinguish operators from c -numbers. In the classical limit the operators $\hat{\psi}$ are substituted by complex numbers ψ satisfying the Hamiltonian equation of motion (Gross-Pitaevskii equation):

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\delta \mathcal{H}(\psi^*, \psi, t)}{\delta \psi^*},$$

where the right hand side denotes a variational derivative. Classically, these equations have to be supplied by definite initial conditions (ψ_0, ψ_0^*) . Within the truncated Wigner approximation – the first order quantum correction to the classical picture – the initial data becomes a distribution $W(\psi_0, \psi_0^*) d\psi_0 d\psi_0^*$. The kernel of this distribution is defined by

$$W_0(\psi_0, \psi_0^*) = \int d\eta^* d\eta_0 \langle \psi_0 - \frac{\eta_0}{2} | \rho | \psi_0 + \frac{\eta_0}{2} \rangle \times e^{-|\psi_0|^2 - \frac{1}{4}|\eta_0|^2} e^{\frac{1}{2}(\eta_0^* \psi_0 - \eta_0 \psi_0^*)}, \quad (\text{B1})$$

where ρ is the initial density matrix of the system. Expectation values of operators $\Omega(\hat{\psi}^\dagger, \hat{\psi})$ at time t are then to be calculated as [30]

$$\langle \Omega(t) \rangle = \int \int d\psi_0 d\psi_0^* W_0(\psi_0, \psi_0^*) \Omega_{cl}(\psi(t), \psi^*(t)), \quad (\text{B2})$$

where $\psi(t)$ is the solution of the classical Gross-Pitaevskii equations of motion with initial condition ψ_0 and $\Omega_{cl}(\psi, \psi^*)$ is the Weyl symbol of the operator Ω . For a normal ordered Ω the latter is defined by

$$\Omega_{cl}(\psi, \psi^*) = \int \int d\eta d\eta^* \Omega(\psi - \eta/2, \psi^* + \eta^*/2) e^{-|\eta|^2/2}, \quad (\text{B3})$$

where $\Omega(\psi, \psi^*)$ is obtained by substitution of operators in $\Omega(\hat{\psi}, \hat{\psi}^\dagger)$ as $\hat{\psi} \rightarrow \psi$ and $\hat{\psi}^\dagger \rightarrow \psi^*$. For example for the number operator $\Omega = \hat{\psi}^\dagger \hat{\psi}$ we get

$$\Omega_{cl} = \int \int d\eta d\eta^* (\psi - \eta/2)(\psi^* + \eta^*/2) e^{-|\eta|^2/2} = \psi^* \psi - \frac{1}{2}. \quad (\text{B4})$$

The same result can be obtained by first symmetrizing number operator with respect to $\hat{\psi}^\dagger$ and $\hat{\psi}$:

$$\hat{\psi}^\dagger \hat{\psi} = \frac{\hat{\psi}^\dagger \hat{\psi} + \hat{\psi} \hat{\psi}^\dagger}{2} - \frac{1}{2} \quad (\text{B5})$$

and then substituting the operators $\hat{\psi}^\dagger$ and $\hat{\psi}$ with the numbers ψ^* and ψ (see Ref. [34]) for more details.

2. Adaption to the spin-boson problem

Our next task is to generalize the TWA to the Hamiltonian (4). To this end, we employ the Schwinger representation of spins through bosons $\hat{\alpha}$ and $\hat{\beta}$:

$$\hat{S}^z = \frac{\hat{\alpha}^\dagger \hat{\alpha} - \hat{\beta}^\dagger \hat{\beta}}{2}, \quad \hat{S}^+ = \hat{\alpha}^\dagger \hat{\beta}, \quad \hat{S}^- = \hat{\beta}^\dagger \hat{\alpha}. \quad (\text{B6})$$

Here, the boson operators α and β satisfy the additional constraint $\hat{n} = \hat{\alpha}^\dagger \hat{\alpha} + \hat{\beta}^\dagger \hat{\beta} = 2S$. The operator \hat{n} commutes with all spin operators which means that the fulfilment of the constraint at an arbitrary time will be conserved for all later times. The option of a purely bosonic representation means that the TWA can readily be generalized to the Hamiltonian (4). Once the TWA has been formulated for the Schwinger bosons, we are at liberty to switch back to spin variables Eqs. (B6). However, at this point, the spins have become classical numbers, defined in terms of the c -number valued boson variables α and β .

The classical equations of motion are given by Eq. (33). The initial density matrix describes a pure spin state, polarized along the z -axis or, in bosonic language, a state with $2S$ α bosons and no β bosons. (For the spin pointing along $-z$ α and β should be interchanged). The corresponding Wigner function then reads [30, 31, 45]:

$$W(\alpha, \alpha^*, \beta, \beta^*) = 2e^{-2|\alpha|^2 - 2|\beta|^2} L_{2S}(4|\alpha|^2), \quad (\text{B7})$$

where $L_N(x)$ is the Laguerre's polynomial of order N . At large S , the latter is strongly oscillatory and the Wigner transform is localized near $|\alpha|^2 = 2S + 1/2$ (see Ref. [30]). So in this case to a very good accuracy (up to $1/S^2$) one can use

$$W(\alpha, \alpha^*, \beta, \beta^*) \approx \sqrt{2} e^{-2|\beta|^2} \delta(|\alpha|^2 - 2S - 1/2). \quad (\text{B8})$$

If we reexpress this distribution function through spins then to the same accuracy we will find

$$W(S_z, S_\perp) \approx \frac{1}{\pi S} e^{-S_\perp^2/S} \delta(S_z - S),$$

where $S_\perp^2 = S_x^2 + S_y^2$. This Wigner function has a transparent interpretation. If the quantum spin points along the z direction, because of the uncertainty principle, the transverse spin components still experience quantum fluctuations so that

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \frac{S}{2},$$

which is indeed the correct quantum-mechanical result. Finally, the distribution of the b -boson represents the Gaussian Wigner function of the vacuum state,

$$W(b, b^*) = 2 \exp[-2b^* b] \equiv W(n) = 2 \exp[-2n],$$

where in the second representation we switched to a representation in terms of the boson number $n = b^* b$.

3. TWA vs quantum solution of linear regime

We here discuss how the solution of the equations of motion (36) obtains information equivalent to that of the full quantum solution of the linear regime. To this end, we notice that the equations afford a solution as (see Eq. (A2))

$$b(t) = \mu b(t_0) + \nu s^-(t_0), \quad s^-(t) = \mu^* s^-(t_0) + \nu^* b(t_0),$$

where at $t \rightarrow \infty$ we have $|\mu|^2 = x$ and $|\nu|^2 = x - 1$ (see Eqs. (A5) and (A6)). If we are interested only in the statistics of the number of bosons we do not need to know the phases of μ and ν . Using the Wigner function (B1) to compute the average and the Weyl symbol (35) for the operators n_b , n_b^2 it is then straightforward to obtain

$$\langle n_b \rangle = x \langle n_b(t_0) \rangle + (x - 1) \langle |s^-(t_0)|^2 \rangle - \frac{1}{2} = x - 1,$$

and the second moment

$$\begin{aligned} \langle n_b^2 \rangle &= -x + \frac{1}{2} + x^2 \langle n_b^2(t_0) \rangle + (x - 1)^2 \langle |s^-(t_0)|^4 \rangle \\ &\quad + 4x(x - 1) \langle n_b(t_0) \rangle \langle |s^-(t_0)|^2 \rangle = 2x^2 - 3x + 1. \end{aligned}$$

It is easy to check that this result conforms with the distribution (14).

APPENDIX C: THE MEANING OF THE FIXED POINTS

Let us consider the time independent version of the model studied in this paper. It is given by the time independent version of the Hamiltonian Eq. (4)

$$\hat{H} = -\frac{\gamma}{2} b^\dagger b + \frac{\gamma}{2} S^z + \frac{g}{\sqrt{N}} (b^\dagger S^- + b S^+). \quad (\text{C1})$$

In the sector where

$$b^\dagger b + S^z = \frac{N}{2},$$

which is the one mostly studied in this paper, this Hamiltonian can be thought of as simply a $N + 1$ by $N + 1$ matrix. In the basis of states $|n\rangle$ where n denotes the boson occupation number, this matrix takes the form

$$\begin{aligned} H_{n',n} &= -\gamma n \delta_{n',n} + \frac{g}{\sqrt{N}} n \sqrt{N - n'} \delta_{n'+1,n} + \\ &\quad \frac{g}{\sqrt{N}} n' \sqrt{N - n} \delta_{n'-1,n} \end{aligned} \quad (\text{C2})$$

(compare with Eq. (9)).

It is easy to evaluate the eigenvalues of this matrix numerically for moderate N .

Now consider a semiclassical version of this Hamiltonian, Eq. (41), which for completeness we reproduce here

$$H = -\gamma n + 2n\sqrt{1 - n} \cos(\phi).$$

Here both the Hamiltonian and the variable n were rescaled according to Eqs. (39) and (40) to ease comparison with the language of section VI, although we need to keep that in mind when comparing it with Eq. (C2) which was not rescaled in any way. Let us minimize this Hamiltonian with respect to n and ϕ . Its minimum is given by the substitution of the fixed point $\phi = 0$, and $n_1(\gamma)$ given by $n_1(\gamma) = 0$ for $\gamma < -2$ and by Eq. (44), or

$$n_1(\gamma) = \frac{12 - \gamma^2 + \gamma\sqrt{12 + \gamma^2}}{18}$$

if $\gamma \geq -2$. To find the energy minimum we substitute $n_1(\gamma)$ into H to find

$$\begin{aligned} E_{\min} &= N \min_{n,\phi} H = 0, \quad \text{if } \gamma < -2 \\ E_{\min} &= -N \frac{36\gamma - \gamma^3 + (12 + \gamma^2)^{\frac{3}{2}}}{54} \quad \text{if } \gamma \geq -2. \end{aligned} \quad (\text{C3})$$

The minimum of the classical Hamiltonian Eq. (C3), together with the eigenvalues of Eq. (C2), are plotted as a function of γ on Fig. 16 for $N = 10$. One can see that the classical minimum closely follows the quantum ground state. Yet Eq. (C3) cannot be the exact ground state of the problem at this value of N since it is not analytic in γ . However, it can be the exact ground state in the limit $N \rightarrow \infty$. Thus we conjecture that Eq. (C3) is the ground state of the Hamiltonian in our problem in the limit $N \rightarrow \infty$. If so, it implies that our problem undergoes a quantum phase transition as a function of γ .

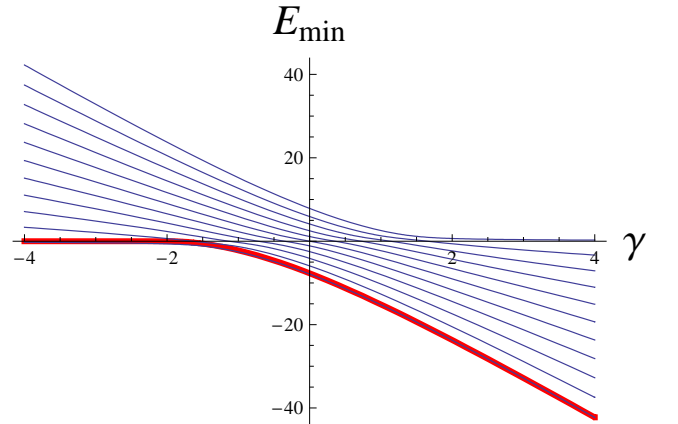


FIG. 16: The energy levels (eigenvalues of the matrix Eq. (C2)) for $N = 10$ are plotted as a function of γ . At the same time, the minimum of the classical Hamiltonian, given by Eq. (C3), is also plotted (red curve). We see that Eq. (C3) closely follows the ground state energy of Eq. (C2).

When γ depends on time, it follows that we drive our system across the quantum phase transition. The existence of the critical trajectory and the singular behavior

of the adiabatic invariants discussed in section VI can be traced to the existence of this transition. The transition in the time independent version of the Dicke model is well known in the literature (see, for example, Ref. [46] and references therein).

APPENDIX D: DERIVATION OF EQ. (62)

Here, just as in the section VI, we use the rescaled variables n and H , according to Eqs. (39), (40). To obtain the value of $\epsilon = \epsilon^*$ at which $w = \pi$, we need to inspect the increment of the angular variable w in the problem where γ changes in time. It satisfies the equation [35]

$$d_t w = \omega + \left(\frac{\partial \Lambda}{\partial I} \right)_{w, \gamma} \lambda,$$

where Λ was defined in Eq. (53). At small λ we neglect the second term to arrive at

$$w(\epsilon) \approx \int_{\epsilon_0}^{\epsilon} d\epsilon' \frac{\pi \sqrt{\epsilon'}}{\lambda \log \left[-\frac{16\epsilon'^2}{H} \right]}. \quad (\text{D1})$$

Here we used Eq. (58), the identity $d\epsilon/dt = 2\lambda$, and $\epsilon_0 > 0$ denotes the moment of time when the system crosses the critical trajectory and switches to trajectories winding about the critical point Eq. (45). This allows us to find w as a function of ϵ , which in turn represents time. To compute this integral, we need to know $H(\epsilon)$. This in turn can be estimated by noting that

$$d_t H = -2\lambda n.$$

Here, n itself is a function of time, which satisfies its equation of motion $d_t n = -2n\phi$. Using Eq. (60) we find

$$n = n_0 e^{-\frac{1}{\lambda} \int_{\epsilon_0}^{\epsilon} d\epsilon' \phi} = n_0 e^{\frac{2}{3\lambda} \left(\epsilon'^{\frac{3}{2}} - \epsilon_0^{\frac{3}{2}} \right)}, \quad (\text{D2})$$

which integrates to

$$H = -n_0 \int_{\epsilon_0}^{\epsilon} d\epsilon' e^{\frac{2}{3\lambda} \left(\epsilon'^{\frac{3}{2}} - \epsilon_0^{\frac{3}{2}} \right)}. \quad (\text{D3})$$

This needs to be substituted into Eq. (D1). We are now in a position to estimate the value ϵ^* corresponding to $w = \pi$. To compute the integrals in Eq. (D1), we need to study the behavior of H . At small $\epsilon - \epsilon_0$, H can be approximated as

$$H \approx -n_0 (\epsilon - \epsilon_0).$$

At larger ϵ , we can estimate it by

$$H \approx -n_0 \frac{\lambda}{\sqrt{\epsilon}} e^{\frac{2}{3\lambda} \epsilon^{\frac{3}{2}}}.$$

Here, ϵ_0 represents the moment in time when the system crosses the critical trajectory. At this point n did

not have a chance to change appreciably from its initial value, $n(\gamma \rightarrow -\infty)$. Thus $n_0 \sim \frac{1}{N}$. This means that to logarithmic accuracy, we may use the approximation $H(\epsilon) \sim -1/N$ in (D1). Other factors, such as $16\epsilon^2$ under the logarithm can also be neglected. This leads to

$$w = \int_{\epsilon_0}^{\epsilon^*} d\epsilon \frac{\pi \sqrt{\epsilon}}{\lambda \log [N]} \sim \frac{2\epsilon^{*\frac{3}{2}}}{3\lambda \log N} \stackrel{!}{\sim} \pi.$$

Solution of this equation for ϵ^* obtains the result (62). Our derivation has been self-consistent in the sense that even at the maximal value ϵ^* ,

$$|H(\epsilon^*)| \sim \frac{\lambda^{\frac{1}{3}}}{\left(\frac{3}{2} \log N\right)^{\frac{2}{3}}} \ll \epsilon^{*2},$$

where again the conditions Eq. (37) were used.

The last thing which remains to be done is to check that Eq. (60) is consistent with the behavior of the system. The equation of motion Eq. (59), together with the condition that $|H| \ll \epsilon^2$, implies

$$2\lambda d_\epsilon \phi = -\epsilon + \phi^2,$$

assuming that $\phi^2 \leq \epsilon$. This is a Riccati equation, which can be brought to the form of the Schrödinger equation by the substitution

$$R = e^{-\frac{1}{2\lambda} \int \phi d\epsilon},$$

to give

$$-2\lambda \frac{d^2 R}{d\epsilon^2} + \epsilon R = 0.$$

This is the equation for the Airy function. It can be investigated using the WKB approximation, which reproduces the ansatz $\phi = -\sqrt{\epsilon}$. Close to $\epsilon = 0$, the WKB approximation breaks down, so this ansatz is no longer correct. However, it is easy to modify relations such as Eq. (D2) and Eq. (D3) by substituting R^2 , with R being the appropriate Airy function, in place of the exponentials in these relations. This does not affect the final answer for ϵ^* , and consequently for I_{final} .

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